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A "good" problem equivalent to the Riemann hypothesis

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Abstract — **We exhibit a sequence c_n such that the convergence of $\sum_{n\geq 1} c_n z^n$ for $|z| < 1$ is equivalent to the Riemann Hypothesis. We argue that this particular RH-equivalent problem is "better" than most, or perhaps every, other RH-equivalent problem devised so far, in the sense that (we prove) there is a tremendous gap in behaviors of the c_n if the RH is true versus if the RH is false.**

Keywords — Riemann zeta function, analyticity, Tauberian theorems, Hejhal, Selberg.

1 A simple attack on the Riemann Hypothesis

The Riemann zeta function is defined by

$$\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s} \tag{1}$$

if re$(s) > 1$, and by analytic continuation for other complex s. $\zeta(s)$ has a simple pole at $s = 1$, but is analytic everywhere else in the complex s plane. The "critical line" is re$(s) = 1/2$. The "Riemann Hypothesis," which is one of the most important open problems in mathematics, is the conjecture that all of the zeros of $\zeta(s)$ in the region re$(s) \geq 1/2$ lie on the critical line. A tremendous amount of information about $\zeta(s)$ and about various statements implied by, implying, or equivalent to the Riemann Hypothesis, or about the known evidence and partial results on it, may be found in the references.

A very simple attack on the Riemann Hypothesis occurred to me. Observe that the **conformal map** $z = 1-1/s$, $s = 1/(1-z)$ makes the region re$(s) > 1/2$ correspond to the unit disc $|z| < 1$. So consider the function[1]

$$F(z) = \ln\left[\frac{z}{1-z}\zeta(\frac{1}{1-z})\right]. \tag{2}$$

This function has been intentionally designed so that the Riemann Hypothesis is equivalent to the statement that $F(z)$ is analytic in the unit disc $|z| < 1$. And this in turn is equivalent to the statement that the Maclaurin series

$$F(z) = \sum_{n\geq 1} c_n z^n \tag{3}$$

converges when $|z| < 1$. (It is easy to see that the c_n are real and that $c_0 = 0$.)

Lo and behold (table 2), c_1, ... c_{25} are positive and strictly decreasing! Obviously, if these two facts were to continue forever, that would immediately imply the Riemann Hypothesis. And furthermore, notice that if you pick 25 random real numbers (e.g. from any probability density symmetric about zero) then the probability that by luck they are going to be positive and decreasing is $2^{-25}/25! \approx 1.9 \times 10^{-33}$. This is smaller than the probability of picking a given air molecule from all the ones in the room. So, as any physicist could tell you (?), it must be the case.

But, in fact, it is not the case, as I discovered when I wrote a program to compute the first 150 c_n. The first increase is $c_{28} \approx 0.022801390 < c_{29} \approx 0.022937613$. Although this is a small increase, it is genuine.

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[1] The real-valued branch of ln is to be used when z is real with $0 < z < 1$; and when z is complex, use the branch arising from continuous variation along the line segment joining 0 and z. This is not always the same as what is got by simply employing the standard version of ln (with a slit along the negative real axis), but it is the same for over 99% of the z in the unit circle, according to my experiments.

But, examining these 150 coefficients, I noticed that although not all of them are decreasing, still, all of them are positive. And the conjecture that all the c_n are positive alone is enough to imply the Riemann Hypothesis. The proof is simple. Assume the RH is false, so some point of nonanalyticity of $F(z)$ exists within the unit circle $|z| < 1$. In that case, since the $c_n > 0$, it would be the case that there must be a singularity of $F(z)$ actually on the *real* interval $(0, 1)$. But, no such singularity exists (obvious from the definition of $\zeta(s)$ for real $s > 1$). QED. Now furthermore, if you pick 150 random numbers, the probability that they, by luck, are all going to be positive, is $2^{-150} \approx 7.0 \times 10^{-46}$. OK, maybe 1.9×10^{-33} was not small enough, but *now*, surely, we have enough confidence. This conjecture must be true, right? Visions of sugarplums danced in my head as I imagined ways to try to prove the c_n are all positive. For example, one can write down various closed forms for the c_n, and then try to prove them positive.

But in fact, this conjecture is also false. I found this out by computing the first 2048 c_n; the first negative one is $c_{156} \approx -0.000139116$. The c_n then monotonically get more negative until reaching a min at $c_{172} \approx -0.005993512$, and then monotonically increase until reaching a max at $c_{217} \approx +0.017704939$ and then monotonically decrease until reaching the most negative coefficient that I know of (indeed, it quite probably is *the* unique most negative coefficient... a conjecture which also would imply the Riemann Hypothesis![2]), $c_{266} \approx -0.008839076$.

Numerical values, rounded to 5 decimal places, for c_1,..., c_{299} are in table 2. These were computed by evaluating $F(z)$ at 2048 points uniformly spaced around a circle of radius 128/129, and then estimating c_n by means of the Cauchy residue theorem with the integrals being evaluated numerically by means of the trapezoidal rule, i.e. a "fast fourier transform." I cannot claim to have a proof that every decimal in the table is correct, but I consider it highly likely because the results were checked by similar computations with different numbers of points and different radii of the circle and while carrying different numbers of decimal places (in the present computation I carried 60 decimal places); and also the first 25 coefficients were computed by an entirely different algorithm (transformation of a known series for $\zeta(s)$) and used as a check. I also used two different languages, `MAPLE` and `MATHEMATICA`. The latter seems to have a far superior (both in speed and accuracy) arbitrary precision implementation of $\zeta(s)$.

Plots of c_n versus n look like a continuous curve which oscillates in a random-looking manner with a rough period (between "maxes") of about 100 and with an apparently gradually decreasing amplitude.

By transforming well known facts about $\zeta(s)$ one may show[3] that

$$F(z) = \frac{K}{1-z} - \ln 2 - \ln \Gamma\left(1 + \frac{1/2}{1-z}\right) + \sum_{\kappa} \ln \frac{\kappa - z}{1-z} \tag{4}$$

where

$$K = \ln(2\pi) - 1 - \frac{\gamma}{2} + \sum_{\kappa}(1-\kappa) = \gamma + \ln\sqrt{\pi} \approx 1.14958 \tag{5}$$

and κ are the locations of the non-real singularities of $F(z)$. Also, $F(z)$ obeys the functional equation[4]

$$F(z) = F(\frac{1}{z}) + \ln(\frac{-z}{\pi}) + \frac{\ln(2\pi)}{1-z} + \ln\sin\frac{\pi}{2(1-z)} + \ln\Gamma(\frac{-z}{1-z}). \tag{6}$$

To conclude:
(1) These quantities (the c_n) deserve more study, since any at most subexponentially increasing bound on them – even a one-sided bound – would imply the Riemann Hypothesis.
(2) It's remarkable how the Riemann zeta function seems to be trying *intentionally* to deceive us!

2 Update: 2005

The preceding part of this paper was written in 1995, titled "Cruel and unusual behavior of the Riemann zeta function," and has been available electronically from my web site ever since. I then forgot about the subject for the next 10 years. And indeed there seemed to be both psychological and mathematical reasons not to have too high an opinion of it.[5]

[2] Because if all coefficients obeyed $c_n > -c$, then $F(z) + c/(1-z)$ would have only positive Maclaurin series coefficients, and then a similar proof to the one above, would imply analyticity for $|z| < 1$.

[3] The sums should be done in order of increasing $|\ln\kappa|$. EQ 4 arises from the Hadamard product formula for the zeta function. The analogue of the usual 4-way symmetry that if ρ is a zeta-zero, then so are $1-\rho$, $\overline{\rho}$, and $1-\overline{\rho}$, is that if $\kappa = 1 - 1/\rho$ is the location of a point of non-analyticity of $F(z)$, then so are $1/\kappa$, $\overline{\kappa}$, and $1/\overline{\kappa}$; this and the pairing of κ and $1/\kappa$ causes the sum in EQ 4 to converge and in fact to cancel out to 0 when $z = 0$. The facts that $F(0) = c_1 = \gamma \approx 0.5772$ and $\Gamma(3/2) = \sqrt{\pi}/2$ force K to take the value given.

[4] This arises from the reflection formula for the zeta function.

[5] Among the psychological reasons were the fact that the same kind of attack on the RH had also been thought of at about the same time by Li [33][5] and even earlier by Keiper [27] (although their treatments seem less simple and transparent). The mathematical reason to be blasé was: over the years many, many problems have been proven to be equivalent to the RH. This is just another such. What is special about it? Indeed, there are many possible variations even just for *our* attack, since the particular function definition in EQ 2 could easily be replaced by an infinite number of alternative possible function definitions which would also work.

But in 2005 my interest was rekindled by communications from Mark W. Coffey (Dept. of Physics, Colorado School of Mines). Coffey has written many published and still-unpublished papers on this, some of which regard my 1995 paper as foundational (or perhaps a better phrase is merely "a good starting point"). He remarked in email:

> This has become a very active (and, I think, very exciting) area of research... Yes, there are other functions to use, but your choice [$F(z)$ in EQ 2] seems to nicely get at the heart of the matter, as it reflects the subdominant behaviour of the Li/Keiper constants. I have a conjecture as to the order of this behaviour...

This news was gratifying. But upon examining the work of Coffey, Li, and others, it seemed to me that insufficiently much theoretical and empirical information was provided about the behavior of the c_n, and the virtues of our particular RH-equivalent problem were not realized clearly enough. I now aim to fill those gaps. We shall

1. Explain why our particular function definition for $F(z)$ is likely to be more useful than many or all alternatives.
2. Explain why this is not just another run of the mill RH-equivalent problem, but instead has some claim to be the *best* one.
3. Prove some interesting theorems about the c_n.
4. Report on the world's largest numerical computation of the c_n, namely for $0 \le n \le 10^5$. (This was accomplished apparently with no additional startling incidences of "unusual cruelty"!) We shall describe a strengthened form of Coffey's conjecture, explain why it is (probably) true, and see that it is entirely compatible with the numerical data.

Why choose this $F(z)$? Consider all possible choices of the form

$$F_{\text{alternative}}(z) = G\left[H(z)\zeta(\frac{1}{1-z})\right] \tag{7}$$

where $G(z)$ is analytic except at $z=0$ and where $H(z)$ is analytic within $|z|<1$ and suitably behaving. First, it is necessary that $H(z)$ behave like z for $|z|$ small, to cancel out the singularity (a simple pole) in $\zeta(s)$ at $s=1$, i.e. $z=0$. The desire to make $H(z)$ have a pole like $1/(1-z)$ at $z=1$ was motivated by the desire to make $F(z)$ look, at $z=1$, as similar as possible to the other singularities of $F(z)$ on the circle $|z|=1$. The simplest possible form of $H(z)$ compatible with these two desires was $H(z)=z/(1-z)$, i.e. the one we used. The only other equally simple contender was $(1-z)z$. But that choice would have unnecessarily complicated EQs 4 and 6. Next, our choice $G(z)=\ln z$ has many advantages. It causes products (there are many in the land of zeta functions) to turn into sums. It is easy to differentiate. Most importantly, logarithmic singularities are the mildest common kind of singularity. Consequently, my top numerical method of finding the c_n's, by using the Cauchy residue theorem $c_n = (2\pi i)^{-1}\oint F(z) z^{-n-1} dz$ (with contour a circle slightly smaller than, and contained in, the unit circle, and not containing any residues of $F(z)$) and the FFT, works very well. If the singularities had been less mild, then the integrand would have had wild behavior and would have been much more difficult to handle numerically.

So in short: the particular choice (EQ 2) of $F(z)$ seems optimal in terms of some combination of (1) validity, (2) simplicity, (3) makes common manipulations and identities simple, (4) makes the c_n maximally amenable to numerical computation.[6]

Why does this RH-equivalent problem seem superior to all others? There are many mathematical problems X that are equivalent to the Riemann Hypothesis RH, in the sense that an affirmative or negative solution to X would imply, and would be implied by, a respectively affirmative or negative solution to the RH [10][11][22][26][42][34][41]. So we want X to somehow be "best."

There have been attempts to find the "best" X before, for various notions of "best." Lagarias [29] evidently thought "best" meant "simplest to state to somebody with the lowest possible education level" and devised a short statement, purely about integers, with no reals, no complex numbers, no calculus, no special functions, and no undetermined constants or "big Os" needed:

$$\text{RH} \Leftrightarrow \sum_{d|n} d \le H_n + \exp(H_n)\ln(H_n), \quad \text{where} \ \ H_n \equiv \sum_{k=1}^{n} \frac{1}{k}. \tag{8}$$

However, a different definition of "best" is "easiest to work with productively to try to settle the RH" and I very much doubt Lagarias's formulation is optimal in *that* sense. Another famous formulation is that the RH is equivalent to various statements that the Möbius function $\mu(n)$ mapping positive integers to $\{-1, 0, +1\}$ "behaves like a random

[6] For some purposes, there may also be grounds for favoring $\exp[-F(z)]$.

function." In particular the RH is equivalent ([42] theorem 14.25C) to the claim that $|\sum_{n=1}^{N}\mu(n)| = O(N^{1/2+\epsilon})$ for each $\epsilon > 0$. There are also many formulations whose goal seems to be to maximize "how different X appears from anything people would normally recognize as the RH." The idea here is to move far away from RH's "home" realms such as number theory and complex analysis, into strange new conceptually-faraway realms (operator analysis, quantum physics, probability theory, dynamical flow systems, approximation theory, algebraic number fields, p-adics...), to allow new ideas to be applied.

The definition of "**best**" I want to focus on here involves a combination of (1) being easy to work with productively (which probably means we need to stay in the "home" area of complex analysis and asymptotics, which is where the most powerful techniques are) and (2) seeming "maximally easy to solve" in the sense that there is a maximum "difficulty gap" working in favor of the RH-prover (or RH-disprover).

Obviously (1) is quite well satisfied by our RH-formulation.

Theorem 1 (Our RH-equivalent problem). *The RH is true if and only if the c_n defined by EQs 2 and 3 are subexponentially bounded (a one-sided $\exp o(n)$ bound in either direction suffices).* (Proofs are in the next section.)

The sense in which our formulation "maximizes" some kind of "difficulty gap" is:

Theorem 2. (Huge difficulty gap in RH-prover's [or disprover's] favor). *If the RH is true, then*

1. *Constants K_1 and K_2 exist (and quite likely $K_1 = c_{266} > -0.00884$ and $K_2 = \gamma = c_1 < 0.57722$) such that $K_1 \le c_n \le K_2$ for all $n \ge 0$.*
2. $\lim_{N\to\infty}(N+1)^{-1}\sum_{n=0}^{N} c_n = 0$. *More generally,* $\lim_{N\to\infty}(N+1)^{-1}\sum_{n=0}^{N}|c_n| = 0$.
3. $\sum_{n=0}^{\infty} c_n/(n+1) = \int_0^1 F(x)\mathrm{d}x$ *converges to a finite limit. So does* $\sum_{n=0}^{\infty}(-1)^n c_n/(n+1) = \int_{-1}^{0} F(x)\mathrm{d}x$*, and more generally,* $\sum_{n=0}^{\infty} e^{2\pi i r n} c_n/(n+1)$ *for any fixed real r.*
4. *But $\sum_{n=0}^{\infty} c_n$ diverges, i.e. has no finite limit. (This particular claim is true regardless of the validity of the RH.) More generally, $\sum_{n=0}^{\infty} e^{2\pi i r n} c_n$ diverges for any fixed r with $0 \le r < 2\pi$ such that $F(z)$ has a singularity at $z = e^{2\pi i r n}$ (there are an infinite set of such r).*
5. *A constant K_3 exists such that $\sum_{n=0}^{\infty}|c_n|^2 = K_3$ (where a 90% confidence interval for the value of this constant is $K_3 = 1.250 \pm 0.031$) and hence $c_n \to 0$.*

The "huge difficulty gap" in a nutshell: to prove the RH, we merely need to prove a 1-sided subexponentially-growing bound on the c_n. This ought to be *far* easier than what is actually true (under the RH), namely, the c_n obey a 2-sided *constant* bound, i.e. do not grow *at all*, and indeed they actually *shrink* to 0 as $n \to \infty$. (This huge gap also aids those trying to disprove RH: any exponential lower bound on the $|c_n|$ suffices – and one exists if RH is false – whereas if the RH is true, then the $|c_n|$ shrink.)

In particular, we claim our RH-equivalent problem is superior to the closely related RH-equivalent problem found by **Li** [33][5]. Li showed that the RH is equivalent to the claim that all

$$\lambda_n = \frac{1}{(n-1)!}\frac{\mathrm{d}^n}{\mathrm{d}s^n}\left[s^{n-1}\ln\xi(s)\right]_{s=1} = \sum_{\rho}\left[1-(1-\frac{1}{\rho})^n\right] \tag{9}$$

(where $n \ge 1$ and the sum is over all non-real zeros ρ of the zeta function) are non-negative.[7] Coffey [9] found the explicit relationship between our c_n and Li's λ_n. Our point is this. Li's is a non-negativity criterion. But we also can phrase our result as a non-negativity criterion: because of our theorem 2 claim 1, our $c_n + K_1$ are non-negative if and only if the RH holds. But we have more. For us it suffices if the c_n grow at most subexponentially. In other words, we do not need non-negativity – a far weaker condition suffices for us. That seems to me to make the present problem clearly superior to Li's problem.

We similarly claim superiority to the RH-equivalent problem devised by M.Riesz in 1916 ([42] §14.32): The RH is true if and only if[8]

$$R(x) \equiv x\sum_{k\ge1}\frac{(-x)^k}{k!\zeta(2k+2)} = x\sum_{n\ge1}\frac{\mu(n)}{n^2}\exp(\frac{-x}{n^2}) \tag{10}$$

obeys $|R(x)| = O(x^{1/4+\epsilon})$ as $x \to \infty$. The reason for our problem's superiority is that $|R(x)| = O(x^{1/2+\epsilon})$ even if the RH is false, so its growth exhibits a much smaller behavior-gap than do our c_n.

[7] $\xi(s) = \xi(1-s) = (s-1)\pi^{-s/2}(\frac{s}{2})!\,\zeta(s)$.

[8] There are also many variants of Riesz's problem, as is clear from Titchmarsh's wonderfully concise derivation. The closed form $\zeta(2k) = 2^{2k-1}|B_{2k}|\pi^{2k}/(2k)!$ may be used in EQ 10 if desired, and the coefficients in $R(x)$'s Maclaurin series may also be simply expressed in terms of the coefficients in the Maclaurin series of $x\cot x$. The first formula for $R(x)$ is the one given by Titchmarsh; the second is got by using $1/\zeta(s) = \sum_{n\ge1}\mu(n)n^{-s}$, where $\mu(n)$ is the $\{0,\pm1\}$-valued Möbius function, in the first and interchanging the order of summation.

Also, our formulation has the advantage that it allows partial progress – researchers can compete with each other to find better bounds on the c_n. In contrast, with Li's formulation, it seemingly is all or nothing.

Now the obvious attack on the RH is: (1) write the c_n is some form particularly amenable to asymptotic analysis (contour integrals? sums? See [7][38][35][2] for ideas). (2) Do that asymptotic analysis well enough to prove a 1-sided subexponential growth bound. (method of stationary phase / saddlepoint method?)

I have not carried out this attack. (In fact, I have barely even scratched its surface.) Therefore I do not have a proof of the RH and I do not have a subexponential upper bound on the c_n. But it is easy to use known results to produce exponential – but just barely exponential – upper bounds on the c_n:

Theorem 3 (Unconditional upper bound on the c_n). *$c_n < K_5 \cdot K_6^n$ where K_5 and K_6 are positive real constants and $K_6 = 1 + 4.5 \times 10^{-25}$. (I have not computed K_5 explicitly with this K_6 although it is mechanically computible and it seems very plausible that $K_5 < 100$. I have performed a smaller version of this computation to find the fully explicit K_5, K_6 pair $K_5 = 4.61$, $K_6 = 1 + 1/401$.)*

Coffey-Smith conjecture (the true asymptotic order of the c_n): *For each $\epsilon > 0$: $|c_n| = O(n^{\epsilon-1/2})$, but for an infinite set of $n > 0$, indeed a positive fraction of them, $c_n > n^{-\epsilon-1/2}$ and (for a different infinite set of $n > 0$ also forming a positive fraction) $c_n < -n^{-\epsilon-1/2}$.*

Plausibility argument for the conjecture:

1. If the "peak" $|c_n|$ are not too atypical (e.g. if the c_n versus n plot continues to "look like a smooth curve"), then the $|c_n|$ must drop at least this fast to allow $\sum |c_n|^2$ to converge (cf. the final claim of theorem 2).
2. If the c_n are modeled as having random sign, then standard ideas about random walks would indicate that $|c_n|$ cannot drop any faster and still permit $\sum c_n$ to diverge (cf. the penultimate claim of theorem 2).
3. Our numerical evidence for $0 < n \le 10^5$ supports the conjecture.

Namely: $|c_n|$ appears to decay more rapidly than $n^{-1/2}$ because[9] the greatest $c_n\sqrt{n}$ known is $c_3\sqrt{3} \approx 0.70477$ and the least known is $c_n\sqrt{n} \approx -0.25738$ when $n = 1867$, despite my calculations of c_n for $0 \le n \le 10^5$. On the other hand, the extreme $|c_n|$ appear to decay less rapidly than $n^{-1/2}\ln(n+1)^{-3}$ because the greatest and least $c_n\sqrt{n}(\ln|n+1|)^3$ known reliably seem to keep increasing and decreasing (respectively) the further one looks, see right half of table 5.

3 Proofs of the theorems

Proof of theorem 1. If there is a 2-sided subexponential bound $|c_n| < B(n)$ where $B(n) \ge 1$ and $\ln B(n) = o(n)$, then the RH follows because the Maclaurin series (EQ 3) converges for all $|z| < 1$ and hence $F(z)$ is analytic throughout this disk and (by the conformal map of the disk to the halfplane re$s > 1/2$ and considering the zeta-function reflection formula) hence the Riemann zeta function has no zeros off the critical line. If there is only a 1-sided subexponential bound, say $-c_n < B(n)$, then the RH still follows because consider $F(z) + G(z)$ where $G(z) = \sum_{n \ge 0} B(n) z^n$. This function has all Maclaurin series coefficients non-negative and real. Therefore its closest point z of non-analyticity to the origin must be on the positive real axis. However, it is obvious from their definitions that $F(z)$ and $G(z)$ both are analytic on $[0, 1)$. So the RH follows.

In the other direction: if the RH is true, then $F(z)$'s Maclaurin series must converge throughout $|z| < 1$. If an infinite set of $n > 0$ existed with $|c_n| > K^n$ for some $K > 1$ (this happens if and only if no subexponential upper bound exists) then the Maclaurin series (as well as the sub-series arising from only these n) diverges at each z with $1/K < |z| < 1$ and indeed the $c_n z^n$ cannot even be bounded. Q.E.D.

Selberg's theorem [40][31][13]: *For any Lebesgue-measurable subset E (whose boundary has areal measure 0) of the xy plane*

$$\lim_{T\to\infty} T^{-1}\text{meas}\left\{T \le t < 2T \ : \ \frac{\ln\zeta(1/2+it)}{\sqrt{(1/2)\ln\ln T}} \in E\right\} = \frac{1}{2\pi}\int\int_E \exp(\frac{-x^2-y^2}{2})\mathrm{d}x\mathrm{d}y. \tag{11}$$

In other words, both the real and imaginary parts of $\frac{\ln\zeta(1/2+it)}{\sqrt{(1/2)\ln\ln t}}$ act asymptotically like independent standard normally distributed random variables.

Hejhal's theorem [18]: *Under a slight extension of the RH[10]: in the $T \to \infty$ limit, the proportion of zeta-zeros*

[9] Indeed, the same sort of evidence makes it appear (less clearly) that the $|c_n|$ decay more rapidly than $n^{-1/2}\ln(n+1)^{-1}$.

[10] Hejhal's extended Riemann hypothesis "H_α": The RH says that all zeta-zeros lie on the "critical line" (with real part 1/2). It is known (regardless of the truth of RH) that the zero with the least positive imaginary part is $1/2 + i14.1347251417...$, and that the number of zeros with imaginary part between 0 and T is asymptotically $(2\pi)^{-1}T\ln[T/(2\pi e)] + O(\ln T)$ when $T \to \infty$. Hejhal's conjectural extension of the RH is that, among the zeros with imaginary part between T and $2T$, for all sufficiently large T, the proportion of spacings Δ between consecutive zeros with $\Delta < x/\ln T$ is upper bounded by Mx^α for some positive constants M and α and for all x with $0 < x < 1$. (In other words, the number of extremely tiny spacings is not ridiculously large.) H_1 is an immediate consequence of the nowadays-usual "GUE hypothesis."

ρ with imaginary part between T and $2T$ which obey

$$a < \left(\frac{1}{2}\ln\ln T\right)^{-1/2} \ln\left|\frac{2\pi\zeta'(\rho)}{\ln(\mathrm{im}\rho/[2\pi])}\right| < b \quad \text{is the same as} \quad \frac{1}{\sqrt{2\pi}}\int_a^b \exp(\frac{-x^2}{2})\mathrm{d}x, \tag{12}$$

i.e. is standard normally distributed (mean=0, variance=1).

History of, and remarks about, the Selberg-Hejhal-Ghosh-Laurinchikas theorems. Atle Selberg proved his theorem in unpublished work in 1949. It was then used by several authors, for example Hejhal [18] to prove his theorem in 1987. This is one reason I feel quite uncomfortable reading [18]. Montgomery [36] surveyed Selberg's accomplishments in the zeta area. Selberg finally got around to publishing a version of his 1949 work in Volume II of his *collected works* [40] in 1991 (see theorems 1 and 2 there, and take $a_n \equiv 1$ and apply the prime number theorem). This publication still is very sketchy, and ends with a promise by Selberg that "a full version with proofs will appear in time." That promise apparently was never fulfilled. In the meantime, Laurinchikas [31], who perhaps was growing impatient, proved the real part of Selberg's theorem – even extended to hold for Dirichlet L-functions instead of merely the Riemann zeta function – in 1987, in the form that

$$\lim_{T\to\infty}\frac{1}{T}\mathrm{meas}\left\{\frac{\ln|L_\chi(1/2+it)|}{\sqrt{(1/2)\ln\ln T}} \le y\right\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y}\exp(\frac{-u^2}{2})\mathrm{d}u. \tag{13}$$

Selberg had proved the imaginary part of his theorem in 1944 (under the RH) and 1946 (unconditionally) in the form[11] that, in the $T \to \infty$ limit,

$$\int_{T-H}^{T} S(t)^{2k}dt = \frac{(2k)!}{k!(2\pi)^{2k}}T(\ln\ln T)^k[1+O(\frac{1}{\sqrt{\ln\ln T}})] \tag{14}$$

for any fixed positive integer k, where $S(t) = \pi^{-1}\arg\zeta(1/2+it)$. Selberg's result had $H = T$, but this later was re-proved and improved by Ghosh [13][16] to hold even if the integral is taken over a considerably shorter interval, namely with any H obeying $T^\alpha \le H \le T$ for any fixed α with $1/2 < \alpha \le 1$, and to allow k to be any real with $k > -1$. We shall not require Ghosh's strengthenings, but it is nice to know they are there. Although it was not noticed immediately, by standard probability theory [28][4] results about power-moments, these results imply (unconditionally) that $S(t)$ is asymptotically normally distributed with mean 0 and variance $2\pi^2\ln\ln t$. Laurinchikas [31] also used the method of power-moments in his proof. The combination of these real and imaginary parts is still not as strong as Selberg's full 2D theorem because a proof of their statistical independence still would be lacking – but fortunately, for the purposes of the present paper, that independence (or lack thereof) does not matter! Finally, we should also note that just because a random variable v is asymptotically normally distributed, does not by itself force certain integrals of it to exist: v could, with some probability that asymptotically goes to 0, behave extremely wildly (e.g. take a value *far* away from anything the normal distribution would predict). However, because these proofs all proceeded via the method of power-moments, we know that in fact, all power-moment integrals of the normally distributed quantity (e.g., mean, variance, etc.) actually *do* exist and converge to the "right" values. As Selberg himself notes, the error term $(\ln\ln T)^{-1/2}$ decreases so slowly that it is infeasible to confirm his theorem by numerical experiments. Nevertheless, Odlyzko ([37], table 2.5.1) studied 175 million zeta-zeros near the 10^{20}th zero and found tolerably good agreement with Selberg's EQ 14 for $2k = 1, 2, \ldots, 8$. Finally, note that, while Hejhal's theorem depends upon a slight extension of the RH, Selberg's theorem holds unconditionally.

Remarks about $S(t) = \pi^{-1}\arg\zeta(1/2+it)$: Since the value of $S(t)$ is also of interest to us because $|S(t)| > 1$ basically means we need to use a nonstandard branch of the natural log in EQ 2, let us make some more remarks about it. Von Mangoldt showed that $|S(t)| = O(\ln t)$ and under the RH [42] $|S(t)| = O(\ln t/\ln\ln t)$. In the other direction Montgomery showed (under the RH) $|S(t)| > \kappa_1\sqrt{\ln t/\ln\ln t}$ on an infinite unbounded set of $t > 0$ and Selberg showed (unconditionally) $|S(t)| > \kappa_2(\ln t)^{1/3}(\ln\ln t)^{-7/3}$ on an infinite unbounded set of $t > 0$ for constants $\kappa_1, \kappa_2 > 0$. The true sup-order is probably $\sqrt{\ln t}$ up to factors of powers of $\ln\ln t$. Under the RH, $S(t)$ is continuous along the critical line, except that it jumps by ± 1 at each zeta-zero. If the RH is false there could be additional jumps.

My computer attempted to find the least $t > 0$ with $|\arg\zeta(1/2+it)| > \pi$, by following the curve of $\zeta(1/2+it)$ in the complex plane from $t = 0$ to $t = 700$ until it crossed the negative real axis. The first such crossing point it found was in $282.454 < t < 282.457$, with a nearby $S(t)$ extreme at $S(282.460) \approx -1.003$. Some other, more

[11] Confusingly, Selberg uses "am" when he means "arg." The branch of arg arising from continuous variation along the vertical line joining 2 and $2+it$ and then along the horizontal line to $1/2+it$ is to be used, where $\arg 2 = 0$. Warning: Selberg's is not exactly the same branch of ln that we use in defining $F(z)$, which would correspond in Selberg's setting to continuous variation along the circle passing through $z = 1$ and orthogonal to the critical line where it hits it. However, Selberg's and our definitions are (1) close enough to the same thing that Selberg's normality theorems still hold with this alternate definition (this is easy to see since the probability-mass that is affected is $O(T^{\epsilon-1})$), and (2) if the RH is true, then they are exactly the same thing.

dramatic, crossings are at $415.599 < t < 415.601$ and $527.6957 < t < 527.6979$. The least t where $S(t) > 1$ obeys $650.667 < t < 650.670$, with a nearby $S(t)$ extreme at $S(650.78) \approx +1.08$.

$|S(t)| > 2$ (necessary to cause "Rosser's rule" to fail) occurs at about $t \approx g_{13999525} \approx 6820051.0$ with a nearby extreme $S(t) = -2.004138$, $t \approx g_{30783329} \approx 14190357.8$ with a nearby extreme $S(t) = -2.002594$, and $t \approx g_{30930927} \approx 14253736.6$ with a nearby extreme $S(t) = +2.050625$. These were found by Brent [8].

For a long time no explicit t with $|S(t)| \geq 3$ was known. But the "zetagrid" RH-confirming multiprocessor project [44] found a positive extreme value $S(t) = +3.0214$ near $t \approx g_{53365784979} \approx 16220609807.6$ and a negative extreme value $S(t) = -3.2281$ near $t \approx g_{67976501145} \approx 20433335722.3$. So far nobody has found any explicit t with $|S(t)| \geq 4$.

Proof of theorem 2's final claim. Assuming the RH, by Parseval's equality and the fact that $c_0 = 0$,

$$\sum_{n \geq 0} |c_n|^2 = \frac{1}{2\pi} \int_0^{2\pi} |F(e^{i\theta})|^2 \mathrm{d}\theta = K_3. \tag{15}$$

Since the integrand in EQ 15 is everywhere non-negative, we do not have to worry about possible oscillatory divergence of the integral; the only worry is that its value might be ∞. Why is the integral finite?

First of all, $F(z)$ has only logarithmic singularities on the unit circle (except at $z = 1$, which is an essential singularity, which, however, looks logarithmic if approached along the real axis from the left) and (under the RH) none within it. It is important to note that logarithmic-power singularities are always integrable, i.e. $\int_{-\epsilon}^{+\epsilon} |\ln x|^p \mathrm{d}x$ is always finite for any fixed $\epsilon, p > 0$. (This is perhaps most easily seen by "looking at the graph sideways" to see the equivalence to the integrability of the inverse function $\exp(x^{-1/p})$ on the positive-infinite real halfline, which is obvious.) Therefore, it is obvious that the integral of EQ 15 on any closed *subinterval* of $(0, 2\pi)$ exists and is finite. It also is obvious that $\int_0^{2\pi} |F(re^{i\theta})|^2 \mathrm{d}\theta$ exists and is finite for any particular r with $0 < r < 1$. The only difficulty arises when we take the limit as the subinterval expands to become $[0, 2\pi)$ itself (or take the limit of the second integral as $r \to 1^-$).

We now claim by Selberg' theorem and its method of proof via power moments, that this integral is finite. This finiteness holds unconditionally; the only part of our proof that requires the RH is the equality of the sum and the integral. Q.E.D.

A deeper look at finiteness of the integral: There are basically two possible finiteness-preventing problems:

1. There are an infinite number of zeros of the zeta function on the critical line, i.e. an infinite number of logarithmic singularities of $F(z)$ on the unit circle, with a unique accumulation point at $z = 1$. Although each singularity makes only a finite contribution to the integral, the summed contribution of all of them might conceivably diverge to ∞.

2. The zeta function on the critical line might assume very large values at places far up the line, i.e. $|F(z)|^2$ on the unit circle might assume very large values as $z \to 1$ even *between* the singularities. This could lead to an infinite integral.

We now explain why neither problem is a problem, in a way that yields a bit more understanding than just saying "because Selberg's theorem says so" (although the latter approach is admittedly both more concise and stronger).

Our conformal transformation of the critical line to the unit circle is very helpful because it "crunches all the singularities a lot narrower" near $z = 1$, lessening the magnitude of the problem tremendously. But it was hardly a problem to begin with thanks to Hejhal's theorem. Observe that as $k \to 0$ with $p \geq 1$ fixed, the value of the integral $\int_{-\epsilon}^{+\epsilon} |\ln(kx)|^p \mathrm{d}x$ is shifted additively by $O(2\epsilon(\ln k)^q)$ for some q with $1 \leq q \leq p$. For us, the appropriate k to use is a rescaled version of the derivative of the zeta function. Our worry is that the sum of all these additive shifts might be infinitely large because there are an infinite number of zeta-zeros ρ with extremely small $|\ln \zeta'(\rho)|$. In view of the "crunching" caused by the conformal tranformation (which tends to multiply k by large values), and in view of the mollifying effect of the ln function, in order for there to be a problem, the $|\ln \zeta'(\rho)|$ would frequently have to be *extremely* small. But, according to Hejhal's theorem, that does not happen. It is not worth going into detail because the gap between the truth (Hejhal's theorem) and the behavior needed to cause a problem is so enormous.

So Hejhal's theorem has taken care of the small zeta function values, in the neighborhood of zeros. We also need to take care of small zeta function values *not* in the neighborhood of zeros – since conceivably the zeta function could get very small (have a magnitude min) even far away from any zero. That does not happen because Anderson [1] proved, under the RH, that there exists a constant T such that starting at any zero, $|\zeta(1/2 + it)|$ increases monotonically to a maximum then decreases monotonically down to the next zero, for all $t > T$.

We also need to worry that the zeta function might suddenly "level off" at a very small value after a very short rise above zero, then stay there. Selberg's theorem rules out that worry.

Finally, we need to handle *large* zeta function values. For that we recall the "Lindelöf hypothesis," a famous known consequence of RH which states that $|\zeta(1/2 + it)| = O(1 + |t|^\epsilon)$ for each $\epsilon > 0$. Indeed under the RH ([42] ch. 14) the right hand side may be improved to $\exp \frac{c \ln t}{\ln \ln t}$ for some fixed $c > 0$.

Even without assuming the RH, Hardy & Littlewood proved $|\zeta(1/2+it)| = O(1+|t|^{1/4+\epsilon})$ and Weyl improved the exponent to $1/6+\epsilon$. See ch.5 of [42] for these and its notes for further improvements; apparently the latest is Huxley's $89/570+\epsilon$ [21]. In view of the mollifying effect of the ln in our F-defining formula (EQ 2) any of these easily suffice to prevent divergence. Q.E.D.

We now attempt to **evaluate the integral numerically** in two ways:

1. "Monte-Carlo method" of trying random θ selected uniformly from $[0, 2\pi)$, see the left half of table 1. The result: with 90% confidence[12] (1.65σ), $K_3 = 1.2505$ to within ± 0.031.

2. Evaluate $\frac{1}{2\pi}\int_0^{2\pi} |F(re^{i\theta})|^2 d\theta$ by the N-point trapezoidal numerical integration rule, for a sequence of (r, N) pairs which increase monotonically to $(1, \infty)$. The result is in the right half of table 1.

sample mean	samp. std. error
1.2265	0.0359
1.3006	0.0370
1.1981	0.0324
1.2768	0.0365
1.2576	0.0182

K_3	N	r
0.2754236	1024	0.721680
0.5211620	2048	0.860840
0.7511443	4096	0.930420
0.9256089	8192	0.965210
1.0411692	16384	0.982605
1.1130223	32768	0.991302
1.1574937	65536	0.995651
1.1860871	131072	0.997826
1.2055086	262144	0.998913
1.2194639	524288	0.999456
1.2299480	1048576	0.999728
1.2380653	2097152	0.999864
1.2393749	4194304	0.999881

Table 1: (left) Four 10000-point Monte-Carlo estimates of K_3 (top) and an independent 40000-point estimate (bottom). The expected value of $|F(z)|^2$ for a random point z on the unit circle $|z| = 1$ is ≈ 1.25 and the standard deviation of $|F(z)|^2$ is ≈ 3.6.
(right) Estimates arising from N-point trapezoidal rule on circles of radius r.

Remarks and conjectures. If computationally effective forms of Selberg's theorem were devised, it would then become possible to compute rigorous arbitrarily close upper and lower bounds on K_3.

Atkinson's mean-square behavior ([42] ch.7; [22] ch.15) that $T^{-1}\int_0^T |\zeta(1/2+it)|^2$ is asymptotically proportional to $\log T$ is very well understood (expansions are available to fairly high order [17]). For further results about the derivative of the zeta function, see [3][34][41][20][32]. I believe that eventually it will be shown that the ln of the zeta function, *and* its first k derivatives, all (when appropriately normalized) act asymptotically (far up the critical line) like independent standard normal random deviates, i.e. yield a $(2k+2)$-dimensional standard normal distribution.

Coffey [9] suggested that perhaps $F(z)$ was univalent in the unit disk, so that the famous *Bieberbach* conjecture (nowadays the de Branges theorem) [15][13] would suffice to show $|c_n| = O(n)$, proving the RH. This hope fails because $F(z)$ is (extremely) not univalent. Specifically, if one draws a near-circular curve $z(t)$ that looks like the unit circle but stays slightly inside it and approaches it closely near $z = 1$, then the curve $F(z(t))$ will wind about wildly and intersect itself many times. (It is enjoyable to get one's computer to draw such curves.) Every such self-intersection is a counterexample to univalency.

Indeed in ([42] ch.11) it is shown that for fixed $a \neq 0$, $1/2 < \alpha < \beta < 1$, the number of points $s = \sigma + it$ in the rectangle $\alpha < \sigma < \beta$, $0 < t < T$ at which $\ln \zeta(s) = a$, is greater than fT for some constant $f > 0$; also each $a \neq 0$ is represented an infinite number of times in $1 < \sigma < 1+\delta$, $0 < t < \infty$ for each $\delta > 0$. In other words, *every* nonzero finite complex value is represented an infinite number of times by $\zeta(s)$ for s in either of these infinitely long rectangles, and zero is also represented an infinite number of times on the line with real part 1/2. Even more amazingly, the $\zeta(s)$ function is "universal" in the sense that for each r with $0 < r < 1/4$ and for *any* function $f(s)$ analytic and zero-free inside the disc $|s| \leq r$ and continuous up to its boundary; for each $\epsilon > 0$ there exists a real

[12] This "confidence" was got from the *sample* standard deviation, hence is not really right. If the Hejhal and Selberg bounds we discuss later in the proof can be made computationally effective, then one could compute an *explicit* upper bound on the variance, at which point genuine, fully rigorous, confidence intervals could be got for Monte Carlo experiments like ours. Also, one then could compute an explicit upper bound on K_3 itself. For example, $K_3 = 2\int_0^\infty \ln|\zeta(1/2+it)|^2(1/4+t^2)^{-1}dt < 2\int_0^\infty \ln(10+t)^2(1/4+t^2)^{-1}dt < 18.9$ would follow (on the RH) if the apparently extremely conservative estimate implied by the first "<" were valid.

[13] Bieberbach-de Branges theorem: If $f(z) = z + a_2z^2 + a_3z^3 + \ldots$ is analytic and univalent in $|z| < 1$, then $|a_n| < n$ unless $f(z)$ is the Koebe function $f(z) = z/(1-\beta z)$ for some constant β with $|\beta| = 1$, in which case $|a_n| = n$.

number $T = T(\epsilon)$ such that $\max_{|s|\le r} |f(s) - \zeta(s + 3/4 + iT)| < \epsilon$. The $\ln \zeta(s)$ function is known [12] to be universal in an even[14] stronger sense: for any function $f(s)$ analytic in $|s| \le 0.05$ and with $\max_{|s|\le 0.06} |f(s)| \le 1$, for any ϵ with $0 < \epsilon < 1/2$ there exists T with $0 < T < \exp\left(\exp\left(10\epsilon^{-13}\right)\right)$ so that $\max_{|s|\le 0.0001} |f(s) - \zeta(s + 3/4 + iT)| < \epsilon$, and the set of such Ts has positive density lower bounded by $\exp\left(-\epsilon^{-13}\right)$.

This universal-approximation behavior of the zeta function is absolutely remarkable and nothing like it is exhibited by any of the usual special functions determined by differential equations, such as $\exp(1/x)$, even in the neigborhood of an essential singularity.[15]

The Abel-Tauber-Littlewood-Landau-Karamata theorem. *Let the Maclaurin series $f(z) = \sum_{n=1}^{\infty} a_n z^n$ converge when $|z| < 1$.*

1. *If the series also converges at $z = 1$, then the value it converges to is the same as $L = \lim_{z\to 1} f(z)$, and L exists.*
2. *If L exists and either $|na_n|$ is bounded, or (in the case of real a_n) na_n is bounded on at least one side by a constant, then the series at $z = 1$ converges, to L.*

History of the Abel-...-Karamata theorem. The first claim was proven by N.H.Abel in 1826. The second (reverse direction) claim was proven by Alfred Tauber in 1897 under the assumption that $na_n \to 0$. G.H.Hardy then coined the name "Tauberian theorems" for results of this second type. J.E.Littlewood in 1911 was able to weaken Tauber's assumption to $|na_n| < M$. E.Landau, in his book [30], then was able to weaken Littlewood's assumption to a one-sided bound (e.g. $na_n < M$) under the assumption the a_n are real. Also, although Littlewood's original proof was long and difficult, a 2-page proof was found by J.Karamata in 1930 [23].

Proof of the rest of theorem 2. First claim (the bounds $K_1 \le c_n \le K_2$): This is now trivial: Because $\sum_n |c_n|^2$ converges, $|c_n|^2 \to 0$ and hence $c_n \to 0$ and hence the c_n are bounded on both sides. Indeed we have the bound $|c_n| < \sqrt{K_3} \approx 1.12$ for all n. Further, if $n > N$ then we may use our explicit computations of the preceding n to get better bounds: $|c_n| < \sqrt{K_3 - \sum_{m=0}^{N} |c_n|^2}$. Unfortunately we do not know the exact value of K_3, but our estimates suffice to make it very plausible that $|c_n| < 0.3$ for all $n > 1000$, which would prove $K_1 = c_1 = \gamma \approx 0.5772$.

Second claim: the average value of the c_n's is zero: $\lim_{N\to\infty}(N + 1)^{-1} \sum_{n=0}^{N} c_n = 0$. This, and more generally, $\lim_{N\to\infty}(N + 1)^{-1} \sum_{n=0}^{N} |c_n| = 0$, are trivial consequences of $c_n \to 0$.

Third claim: $\sum_{n=0}^{\infty} c_n/(n + 1) = \int_0^1 F(x)\mathrm{d}x$ converges to a finite limit. To see this, we consider the function $\int_0^z F(x)\mathrm{d}x$ whose Maclaurin series coefficients are $c_n/(n + 1)$. By considering the boundedness of the c_n we realize that the Littlewood-Karamata Tauberian theorem may be applied to see that $\sum_{n=0}^{\infty} c_n/(n + 1) = \int_0^1 F(x)\mathrm{d}x$. It is obvious from the definition of $F(z)$ that this integral is finite (the singularity at $z = 1$ is logarithmic and hence integrable).

More generally: $\sum_{n=0}^{\infty}(-1)^n c_n/(n + 1) = \int_{-1}^{0} F(x)\mathrm{d}x$ and $\sum_{n=0}^{\infty} e^{2\pi irn} c_n/(n + 1)$ for any fixed real r are proven in the same manner, but the integral now is along the line segment from 0 to $e^{2\pi ir}$. These integrals all are finite under the RH since the singularities at their endpoints (if any) are logarithmic.

Fourth claim: But $\sum_{n=0}^{\infty} c_n$ diverges, i.e. has no finite limit, regardless of the validity of the RH.

If the RH is false, then from theorem 1 we know that an infinite subsequence of the $|c_n|$ increase unboundedly and exponentially, assuring divergence. If the RH is true, then $\sum_{n=0}^{\infty} c_n$ cannot converge to any finite value V, because then by Abel's theorem, we would have $V = \lim_{z\to 1} F(z)$, but in fact this limit is divergent. More generally, the same reasoning shows that $\sum_{n=0}^{\infty} e^{2\pi irn} c_n$ diverges, for any fixed r with $0 \le r < 2\pi$ such that $F(z)$ has a singularity at $z = e^{2\pi irn}$ (there are an infinite set of such r). Q.E.D.

Proof of theorem 3. The key lemma we shall use is: If a function $f(z)$ is analytic throughout a circular disk containing $z = 0$ and the maximum value of $|f(z)|$ on the circle's boundary is M, and the minimum value of $|z|$ on the circle's boundary is r, then f's Maclaurin series coefficients obey $|c_n| \le M/r^n$. This is an immediate consequence of the Cauchy residue theorem.

A non-rigorous (but still quite convincing) maximization of $|F(z)|$ on the circle $|z| = 0.9998808$ finds $K_5 = M < 9.1$ and $K_6 = 1/r < 1.00012$.

Suppose we know that the zeta function has no zeros in the critical strip (aside from those on the critical line) with imaginary parts between 0 and H. Then the circle that passes through the 3 points $1 + iH$, $1 - iH$, and $1/2$ is zero-free. Upon applying our conformal transformation, this circle is mapped to a different circle C, namely the one with center $-1/(4H^2 + 2)$ and tangent to the unit circle from inside it. The theorem immediately follows with $K_5 = M$ where M is the maximum value of $|F(z)|$ on C, $r = 1 - 2/(4H^2 + 2)$, and $K_6 = 1/r = 1 + 1/(4H^2 + 1)$.

By using $H = 10$ and a rigorous 1D global maximizer,[16] I found $K_5 = 4.61$, $K_6 = 1 + 1/401$.

[14] And consequently our $F(z)$ function is also universal in $|z| < 1$ in an appropriate sense.

[15] Because: all derivatives of the usual transformations of the the usual functions at any point are determined by the first few of them at that point, preventing universality.

[16] Henrici's complex "disk arithmetic" may be employed for this purpose [19].

Brent et al. [8] verified the RH for the first 200000001 zeros. The same sort of approach was used by the "zetagrid" distributed computing project of S.Wedeniwski [44] to reach about the 10^{12}th zero in about 700 years of computer time.

A far superior algorithm, asymptotically, is based on Odlyzko and Schönhage's wonderful method [38] for simultaneous approximate evaluation of the zeta function at many points on the critical line. This was implemented by Xavier Gourdon and Patrick Demichel [14] who reached the 10^{13}th zero in only about 1.5 computer-years (then did it again, using the second run, with slightly different parameters, as a check on the first). This proves the RH with $H = 2.38 \times 10^{12}$. Consequently we may take $K_6 = 4.5 \times 10^{-25}$.

My numerical experiments suggest that the maximum M always occurs on the positive real axis. If so, there is no need to search for the maximum, one can just go right to it, and $M = F(1-2/(4H^2+2)) \approx \ln(2H^2)$. This would lead to $K_5 < 58$. QED.

4 Numerical calculation of the c_n for $0 \le n \le 10^5$

My original calculations of the c_n in 1995 employed MAPLE. I thank Henry Cejtin for translating my MAPLE program into MATHEMATICA, which (at least at that time) had a far faster and more accurate arbitrary-precision zeta-function routine.

When I revisited the problem in 2005, I translated the program into C and added some improvements. The resulting program is far faster. It is available electronically from my web page
(`http://math.temple.edu/~wds/homepage/works.html` #33).
It works as follows. First, we choose a radius value r with $0 < r < 1$ and a positive number p. Second, we compute $F(re^{i\theta})$ for $2^{p+1}\pi\theta = 0, 1, 2, \ldots, 2^{p-1}$. This computation is performed with the aid of Borwein's algorithm [6] for evaluating the zeta function. Third, by using a complex→real FFT [39], we compute approximations to all $c_n = r^{-n} \int_0^{2\pi} F(re^{i\theta}) e^{-in\theta} d\theta$ for all $n = 0, 1, 2, \ldots, 2^p - 1$ simultaneously in $O(p2^p)$ steps by using the trapezoidal rule for numerical integration. Fourth, only c_n for $n = 0, 1, 2, \ldots, 0.07 \cdot 2^p$ or so are kept and the rest are discarded as having too large numerical integration errors.

It is necessary both to choose p and r, and to code Borwein's algorithm, with some care to try to make sure that IEEE 64-bit real arithmetic is not overly stressed. Also, when computing the natural log, it is necessary to be careful to get the right branch. This was accomplished in my program by adding the integer multiple of $2\pi i$ to the log which caused it to be closest to the previously computed log (as we go around the circle of integration). This admittedly could conceivably return the wrong answer.[17] However, whenever $2^k \le 2^{22}$ integration points were used (which was as far as I went) with the circle radii I selected, it was found that *no* branch adjustments whatever were performed (much to my surprise), i.e. the standard unadjusted log would have worked just as well since its slit was never crossed.

One may compute the first 10^5 coefficients c_n in a few minutes, but going much beyond that seems impossible unless the program is redone to employ reals wider than 64 bits. If that were done, then 10^8 coefficients ought to be obtainable on a machine with enough memory (or if an out-of-core FFT routine were used). Sanity checks include: comparisons with other calculations arising from different r and p, different zeta-evaluation algorithms, and comparisons with known exact values of the c_n with small n (see table 3).[18] In principle by use of "disk arithmetic" [19] in combination with its real analogue "interval arithmetic" it would be possible to get rigorous bounds on every c_n, but I have not implemented that idea.

5 Acknowkedgement

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References

[1] R.J.Anderson: On the function $Z(t)$ associated with the Riemann Zeta Function, J.Math.Anal.Applic. 118,2 (1986) 323-340.

[2] Luis Baez-Duarte: On Maslanka's representation for the Riemann zeta-function, arXiv.org:math/0307214 (16 July 2003).

[3] B.C.Berndt: The number of zeros of $\zeta^{(k)}(s)$, J. London Math. Soc. (2) 2 (1970) 577-580.

[17] Especially if the RH is false! However, the RH is true in the region my computer has explored so far, and a long way further, so this has not been a concern.

[18] Another sanity check (which I did not implement) would be to rotate the integration points on the $|z| = r$ circle by random angles. This would, however, require a full complex→complex FFT.

$a\backslash b$	0	1	2	3	4	5	6	7	8	9
0		.57722	.48344	.40690	.34390	.29165	.24805	.21146	.18061	.15451
1	.13238	.11359	.09762	.08406	.07256	.06284	.05465	.04779	.04208	.03737
2	.03353	.03045	.02802	.02615	.02477	.02380	.02318	.02287	.02280	.02294
3	.02324	.02367	.02419	.02479	.02543	.02609	.02675	.02739	.02801	.02858
4	.02910	.02956	.02994	.03024	.03047	.03060	.03065	.03061	.03048	.03026
5	.02996	.02957	.02909	.02854	.02792	.02722	.02646	.02564	.02477	.02385
6	.02289	.02189	.02087	.01982	.01876	.01770	.01662	.01556	.01450	.01346
7	.01244	.01144	.01048	.00956	.00868	.00784	.00706	.00632	.00565	.00503
8	.00447	.00398	.00355	.00318	.00289	.00266	.00249	.00240	.00236	.00240
9	.00249	.00264	.00286	.00312	.00345	.00382	.00423	.00469	.00519	.00572
10	.00629	.00688	.00749	.00812	.00877	.00942	.01008	.01073	.01139	.01203
11	.01266	.01327	.01387	.01443	.01497	.01547	.01594	.01637	.01676	.01710
12	.01740	.01765	.01785	.01799	.01808	.01812	.01811	.01803	.01791	.01772
13	.01749	.01720	.01686	.01647	.01602	.01554	.01500	.01443	.01381	.01316
14	.01247	.01176	.01101	.01025	.00946	.00866	.00784	.00702	.00619	.00536
15	.00453	.00371	.00290	.00211	.00134	.00059	−.00014	−.00084	−.00150	−.00213
16	−.00271	−.00326	−.00376	−.00422	−.00463	−.00498	−.00529	−.00554	−.00574	−.00588
17	−.00596	−.00599	−.00597	−.00588	−.00574	−.00555	−.00530	−.00500	−.00464	−.00424
18	−.00379	−.00329	−.00275	−.00217	−.00156	−.00090	−.00022	.00049	.00123	.00199
19	.00277	.00356	.00436	.00517	.00598	.00679	.00760	.00840	.00918	.00995
20	.01070	.01143	.01213	.01280	.01344	.01405	.01461	.01514	.01562	.01605
21	.01644	.01678	.01707	.01730	.01749	.01761	.01769	.01770	.01767	.01757
22	.01743	.01722	.01697	.01666	.01630	.01589	.01544	.01494	.01439	.01381
23	.01318	.01252	.01183	.01110	.01035	.00957	.00878	.00796	.00714	.00630
24	.00545	.00461	.00376	.00291	.00208	.00125	.00043	−.00036	−.00114	−.00189
25	−.00262	−.00331	−.00398	−.00461	−.00520	−.00576	−.00627	−.00674	−.00717	−.00755
26	−.00788	−.00816	−.00840	−.00858	−.00872	−.00880	−.00884	−.00882	−.00876	−.00865
27	−.00849	−.00828	−.00803	−.00773	−.00739	−.00701	−.00659	−.00613	−.00565	−.00512
28	−.00457	−.00400	−.00340	−.00277	−.00213	−.00148	−.00081	−.00013	.00056	.00125
29	.00194	.00263	.00331	.00399	.00465	.00530	.00594	.00656	.00716	.00773

Table 2: Values of c_{10a+b} to 5 decimal places.

[4] P.Billingsley: Probability and measure, Wiley (3rd ed.) 1995.

[5] E.Bombieri & J.C.Lagarias: Complements to Li's criterion for the Riemann hypothesis, J.Number Theory 77,2 (1999) 274-287.

[6] P.Borwein: An efficient algorithm for the Riemann zeta function, Canad. Math. Soc. Conf. Proc. 27 (Amer.Math.Soc. 2000, held at Limoges 1999) 29-34. `http://www.cecm.sfu.ca/preprints/1995pp.html` #95:043.

[7] J.M.Borwein, D.M. Bradley, R.E. Crandall: Computational Strategies for the Riemann Zeta Function, J. Comput. Appl. Math. 121, 1-2 (2000) 247-296.

[8] R.P. Brent, J. van de Lune, H.J.J. te Riele, D.T. Winter: On the Zeros of the Riemann Zeta Function in the Critical Strip, Math's. of Computation 33, 148 (1979) 1361-1372; 39,160 (1982) 681-688; 41,164 (1983) 759-767; 46,174 (1986) 667-681.

[9] Mark W. Coffey: Polygamma theory, the Li/Keiper constants, and validity of the Riemann Hypothesis, manuscript, 5 Feb. 2005.

[10] J. Brian Conrey: The Riemann Hypothesis, Notices of AMS 50,3 (March 2003) 341-352.

[11] Harold M. Edwards: Riemann's zeta function, Academic Press 1974.

[12] Ramūnas Garunkštis: The effective universality theorem for the Riemann zeta function, Proc. Session in Analytic Number Theory & Diophantine Equations, Bonner Math. Schriften 360, Univ. Bonn, Bonn, 2003 (21 pages). `http://www.mif.vu.lt/~garunkstis/preprintai/effective.pdf`

n	c_n	exact
0	0	0
1	0.57721566490153286060651209008240243	γ
2	0.48344254848135024830791039777968164	$(1-\gamma/2)\gamma-\gamma_1$
3	0.40689897607223193394331144650482890	$(1-\gamma+\gamma^2/3+\gamma_1)\gamma-2\gamma_1+\gamma_2/2$
4	0.34389703296781448149657017705425033	
5	0.29165370003943353069979075344418507	
6	0.24804972120203626650995706617365573	
7	0.21145583431983466073561173839154791	
8	0.18060696801492895703186513892928415	
9	0.15451071186569955316461332520898556	
10	0.13238036836962940606333805531494826	

Table 3: 35 decimal-place (rounded) values of c_n for $n=0,\dots,10$, computed from exact expressions in terms of the Stieltjes constants $\gamma_n = \lim_{m\to\infty}\left[-\ln(m)^{n+1}/(n+1)+\sum_{k=1}^{m}\ln(k)^n/k\right]$ where $\gamma_0=\gamma$ is the Euler-Mascheroni constant. The general exact expression for c_n in terms of the γ_ℓ has been given by M.W.Coffey [9] and ultimately arises from the well known expansion $\zeta(s)=(s-1)^{-1}+\sum_{n\ge 0}(1-s)^n\gamma_n/n!$.

[13] Amit Ghosh: On the Riemann zeta function – mean value theorems and the distribution of $|S(t)|$, J. Number Theory 17,1 (1983) 93-102

[14] Xavier Gourdon: The 10^{13} First Zeros of the Riemann Zeta Function, and Zeros Computation at Very Large Height, 24 Oct. 2004 `http://numbers.computation.free.fr/Constants/Miscellaneous/zetazeros1e13-1e24.pdf`.

[15] Arcadii Z. Grinshpan: The Bieberbach conjecture and Milin's functionals. Amer. Math. Monthly 106,3 (1999) 203-214.

[16] L. Groenewald: On the means of the argument of the Riemann zeta-function on the critical line, Acta Math. Hungar. 58, 1-2 (1991) 25-29.

[17] James Lee Hafner & A. Ivic: On the mean square of the Riemann zeta function on the critical line, J.Number Theory 32,2 (1989) 151-191.

[18] Dennis A. Hejhal: On the distribution of $\ln|\zeta'(\frac{1}{2}+it)|$, pp. 343-370 in *Number Theory, Trace Formulas, and Discrete Groups,* (eds. K.E.Aubert, E.Bombieri, D.M.Goldfeld) Proc. Selberg Sympos. (Oslo, July 1987) Academic Pr. Boston MA 1989. reviewed in Zbl 0665.10027 by D.R.Heath-Brown and in MR 0993326 (90j:11085) by H.L.Montgomery; also discussed in [20].

[19] Peter Henrici: Applied and Computational Complex Analysis (3 Volumes), Wiley 1996.

[20] C.P.Hughes, J.P.Keating, N.O'Connell: Random matrix theory and derivatives of the Riemann zeta function, Proc. Math'l, Phys'l and Engineering Science (former name: Proc. Royal Soc. London ser. A) 456, 2003 (2000) 2611-2627.

[21] M.N. Huxley: Exponential sums and the Riemann zeta function IV, Proc. London Math. Soc. (3) 66,1 (1993) 1-40.

[22] Aleksandar Ivic: The Riemann zeta-function, Wiley 1985.

[23] Jovan Karamata: Über die Hardy-Littlewoodschen Umkehrungen des Abelschen Stäigkeitssatzes, Mathematische Zeitschrift 32,2 (1930) 319-320.

[24] Anatolii Karatsuba: Basic analytic number theory, translated from the Russian by Melvyn B. Nathanson, Springer-Verlag, 1993.

[25] Anatolii Karatsuba: Complex analysis in number theory, CRC Press, 1995.

[26] A.A. Karatsuba & S.M. Voronin: The Riemann zeta-function, translated from the Russian by Neal Koblitz, Walter de Gruyter, 1992.

[27] J.B.Keiper: Power series expansions of Riemann's ξ function, Math's. of Computation 58, 198 (1992) 765-773.

[28] M. Kendall & A. Stuart: The Advanced Theory of Statistics. (Multivolume series currently published by Arnold Publishers.)

[29] J.C.Lagarias: An elementary problem equivalent to the Riemann hypothesis, Amer. Math. Mon. 109,6 (2002) 534-543.

[30] Edmund Landau: Darstellung und Begründung Ergebnisse der Funktionentheorie, Springer, 1916, second edition 1929, third edition 1986 edited and supplemented by Dieter Gaier.

[31] A.Laurinchikas: A limit theorem for Dirichlet L-functions on the critical line, Lithuanian Math. J. 27,4 (1987) 316-324.

[32] A.A. Lavrik: Discrete means of derivatives of the Riemann zeta function on the critical line, Izv. Akad. Nauk UzSSR ser. Fiz.-Mat. Nauk #4 (1988) 25-31, 104. Reviewed by A.Ivic in MR0979382 (90d:11098) and Zbl 0669.10063.

[33] Xian-Jin Li: The positivity of a sequence of numbers and the Riemann hypothesis, J.Number Theory 65 (1997) 325-333.

[34] N.Levinson & H.L.Montgomery: Zeros of the Derivatives of the Riemann Zeta-Function, Acta Math. 133 (1974) 49-65.

[35] Krzysztof Maślanka: Hypergeometric-like representation of the zeta-function of Riemann, math-ph/0105007v1, 4 May 2001.

[36] H.L. Montgomery: Selberg's work on the zeta function, pp. 157-168 in *Number Theory, Trace Formulas, and Discrete Groups,* (eds. K.E.Aubert, E.Bombieri, D.M.Goldfeld) Proc. Selberg Sympos. (Oslo, July 1987) Academic Pr. Boston MA 1989.

[37] A.M. Odlyzko: The 10^{20}th zero of the Riemann zeta function and 175 million of its neighbors. Unpublished book, 1992. Available at `http://www.dtc.umn.edu/~odlyzko/unpublished/index.html`.

[38] A.M. Odlyzko & A. Schönhage: Fast algorithms for multiple evaluations of the Riemann zeta function, Trans. Amer. Math. Soc. 309 (1988) 797-809.

[39] W.H.Press, B.P.Flannery, S.A.Teukolsky, W.T.Vetterling: Numerical recipes in C, Cambridge Univ. Press 1992.

[40] Atle Selberg: Old and new conjectures about a class of Dirichlet series, pp. 47-65 in his *Collected Work* Vol. II, Springer, New York, 1991. Also, many of Selberg's hard-to-obtain Norwegian papers are in volume I (Springer 1989). The original citations for the ones reprinted on pp. 179-203 and 214-280 are: On the remainder in the formula for $N(T)$, the number of zeros of $\zeta(s)$ in the strip $0 < t < T$, Avhand. Norske Vid. Akad. Oslo I. Mat.-Naturvid. Kl. #1 (1944) 1-27 and Contributions to the theory of the Riemann zeta-function, Archiv for Math. og Naturvidenskob B, 48 (1946) 89-155.

[41] A. Speiser: Geometrisches zur Riemannschen Zetafunktion, Math. Annalen 110 (1934) 514-521.

[42] E.C. Titchmarsh: The theory of the Riemann zeta-function, 2nd ed., revised by D.R. Heath-Brown, Oxford Clarendon Press 1986.

[43] S.M. Voronin: A theorem on the "universality" of the Riemann zeta-function, Math. USSR-Izv. 9,3 (1975) 443-453.

[44] S.Wedeniwski: "Zetagrid" collaborative computing project, `http://www.zetagrid.net`.

n	c_n min	n	c_n max
28	0.02280139009458	46	0.03065351260530
88	0.00236364969117	125	0.01812255037514
171	−0.00599351249263	217	0.01770493871324
266	−0.00883907608469	315	0.01253988255037
363	−0.00131379139182	399	0.00352098769716
437	−0.00155950380580	484	0.00792412206467
532	−0.00265011369606	576	0.00586407479860
625	−0.00453989662342	683	0.00608780194195
792	−0.00876015366627	855	0.00985797074639
964	−0.00336332519566	1019	0.00389688597804
1066	−0.00153818488646	1115	0.00440021905772
1164	0.00005467880305	1196	0.00128589755472
1242	−0.00163007107445	1287	0.00135512473644
1325	−0.00066526335624	1376	0.00426339234538
1428	−0.00136942894120	1476	0.00186609695671
1517	0.00064404768880	1535	0.00079246932437
1590	−0.00292177900204	1647	0.00389781719388
1698	−0.00069767736858	1796	0.00309995685116
1867	−0.00595669821220	1995	0.00546016404804
2121	−0.00319014471526	2181	0.00199295480007
2226	−0.00027911074669	2271	0.00160331956098
2314	0.00032723408933	2350	0.00102857036774
2387	0.00020014426637	2436	0.00267830060041
2497	−0.00296619173065	2599	0.00215990443289
2663	−0.00229393735603	2722	0.00194668557918
2760	0.00107257403797	2792	0.00159322788876
2847	−0.00068292293008	2893	0.00053831453272
2921	0.00020096863083	2967	0.00168515230893
3030	−0.00273106705503	3171	0.00341167577060
3190	0.00334465535764	3218	0.00359528973024
3298	−0.00410033760978	3412	0.00101623511632
3457	−0.00022053626542	3510	0.00215512601679
3563	−0.00020649212183	3608	0.00078867711178
3635	0.00058636420848	3681	0.00158013130285
3742	−0.00057707494222	3763	−0.00044296616850
3815	−0.00209543484163	3879	0.00110068098692
3916	0.00051058601068	3953	0.00091930447108
3987	0.00057340355542	4033	0.00161809380293
4089	−0.00096464225022	4143	0.00112397812430
4252	−0.00063700407451	4303	0.00099555682437
4360	−0.00167782325585	4480	0.00206540380103
4545	−0.00024450788597	4566	−0.00015192611164
4611	−0.00095757190767	4670	0.00142737029257
4721	0.00010479910528	4765	0.00057183361261
4779	0.00055284027929	4827	0.00127042815273
4895	−0.00201872728232	4938	−0.00118356283676
4970	−0.00155649823678	5094	0.00357052272782
5166	−0.00064943875440	5220	0.00050367702005
5238	0.00046722765447	5266	0.00059542307428
5345	−0.00239302084422	5382	−0.00213200822984

Table 4: Mins and maxes of c_n for $n \le 5390$. (We say c_n is a "min" if $c_n < c_{n+1}$ and $c_n < c_{n-1}$, and a "max" if $c_n > c_{n+1}$ and $c_n > c_{n-1}$.) There are 50 maxes here for an average spacing ("approximate period") of $5381/50 \approx 108$ between maxes. This period-length appears at first glance to stay roughly constant forever, but a more precise look (see left half of table 5) up to $n = 10^5$ suggests slow growth of the average max-spacing among $c_1, \dots, c_n$, e.g. perhaps period-length$\approx 55 + 6.6 \ln n$, although it is difficult to be certain of that law because of the slowness of the growth and the presence of considerable "noise." Coefficients thought accurate to unit last place, but that is not proven.

#maxes	n	period	fit
11	1115	101.27	101.31
20	2181	109.00	105.74
39	4143	106.21	109.97
74	8307	112.24	114.56
138	16461	119.28	119.08
265	33015	124.58	123.67
505	65609	129.92	128.20
764	100093	131.01	130.99

max#	type	n	c_n
0	max	1	+0.5772156649015329
	min	28	+0.0228013900945776
1	max	46	+0.0306535126052961
	min	88	+0.0023636496911747
2	max	125	+0.018122550375142
	min	171	-0.005993512492634
3	max	217	+0.017704938713244
	min	266	-0.008839076084687
4	max	315	+0.012539882550375
	min	625	-0.004539896623417
8	max	683	+0.006087801941950
	min	792	-0.008760153666268
9	max	855	+0.00985797074639
	min	1867	-0.00595669821220
19	max	1995	+0.00546016404804
30	max	3218	+0.00359528973024
	min	3298	-0.00410033760978
47	max	5094	+0.00357052272782
	min	9947	-0.00194200879367
	min	14924	-0.00140007340125
135	max	15788	+0.00146729839833
171	max	20663	+0.00119506685908
	min	20888	-0.00139689995959
228	max	28209	+0.0009962980283
241	max	30077	+0.0009560499787
279	max	35484	+0.0008416255356
	min	47010	-0.0007627145359
463	max	60105	+0.0006034754003
	min	78889	-0.0005415718701
610	max	79302	+0.0005948229714
	min	87180	-0.0005343783046

Table 5: (left) The number m of "maxes" among $c_1, \ldots, c_n$, where c_n is the last max, and where we do not count c_1 as a max. The approximate "period" is $(n-1)/m$. The "fit" is $55 + 6.6 \ln n$.
(right) All spectacular c_n maxes and mins for $0 \le n \le 10^5$. A max or min c_n is "spectacular" if that n yields a new record high (or low) of $c_n n^{1/2} \log(n+1)^3$. Coefficients are thought accurate to unit last place but that is not proven.