"Asset voting" scheme for multiwinner elections

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Abstract —

Most democracies involve elections in which some large number of voters somehow choose W winners from among N candidates, $0 < W < N$. It is desired that those winners "represent" those voters well, and (simultaneously) that the voters tend to "prefer" them to the nonwinners. Unfortunately, these requirements can conflict and are vague. We propose "asset voting," a new voting scheme which seems subjectively superior to all previously proposed multiwinner schemes I know. Objectively, it has 12 advantages (which we state reasonably explicitly), whereas I know no competitor with all 12. An experimental comparison of asset voting with other schemes in the single winner case shows that it is sometimes superior to, and other times inferior to, the previously undisputed champion "range voting." The best available competitors to asset voting in the multiwinner case are forms of Hare's Single Transferable Vote equipped with the "Droop quota."

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1 Asset voting (and some variants)

Let there be N candidates from whom the voters have to choose W winners, $0 < W < N$. This is sometimes called the problem of "electing a committee."

"Asset voting," our new election scheme:
Each voter's vote is an N-tuple of nonnegative reals summing to 1. These vote-vectors are added to get each of the N "vote-totals." Now, we do *not* just take the greatest W coordinates of the sum-vector to determine the winners. Instead we enter a "negotiation" stage, in which any subset of the candidates is allowed to agree to re-allocate its votes among themselves. After all such re-allocations have ceased, *then* the owners of the top W vote-counts are the winners.

Example: Suppose there are $N = 4$ candidates A, B, C, D and 5 voters, and we wish to select $W = 2$ winners. Suppose the votes are as in table 1.1. If no negotiation were to take place, then C and D would be the co-winners. However, suppose instead that B agrees to sacrifice his votes by awarding them to A. In that case A, now with 1.8 votes, and D (still with 1.9) would (regardless of what C does) be the co-winners.

voter\canddt	A	B	C	D
#1	.7	.3	0	0
#2	.3	.5	.2	0
#3	0	0	1	0
#4	0	0	.1	.9
#5	0	0	0	1
total	1	.8	1.3	1.9

Figure 1.1. Voting example. ▲

A perceived disadvantage of the proposed scheme might be the unconstrained character of the negotiations. Voters might become unhappy about the possibility of mysterious, perhaps partially secret, deals. For example, suppose all the candidates walk into a smoke-filled room, then walk out an hour later to announce the W winners, which seem completely unrelated to the votes, and then everybody refuses to say how those W were chosen. To overcome that, here are four possible more constrained **variant schemes:**

1. We could demand the following **bottom-up negotiation procedure:**

1. Find the candidate who got the fewest votes.
2. Eliminate him and ask him to distribute his votes among the (not-yet-eliminated) candidate(s) of his choice in any way he pleases.
3. Repeat until only W candidates remain. They are the winners.

2. Same except that "sure winner" candidates with more than $1/(W + 1)$ of the votes are also questioned and asked if they wish to donate any of their votes to anybody else (which they may well wish to do, since they have "excess" votes).

3. We could also further constrain the negotiations by demanding, in step 2 of the procedure of variant 1, that each eliminated candidate award *all* of his votes to a *single* not-yet-eliminated candidate of his choice.

4. Optionally, we could modify variant 3 by permitting the eliminated candidate to keep all his votes for himself, "taking them to the grave."

These variants also have the advantage that they definitely *terminate* in N or $N-W$ steps. In all of these variant schemes, the full step-by-step description of who donated how much to whom, would be announced to the public as part of the announcement of the election results. Also, it would be best if the terms of any deals the candidates make among themselves, were made public too. In most cases the candidates

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themselves would wish to do that, but in cases when they do not, the publication requirement seems unenforceable.

Of these, I presently prefer variant #2, with the candidate chosen each negotation round for questioning being the one with $> 1/(W+1)$ of the votes (but the least votes subject to this) if one exists, otherwise the one with the least votes, both of these subject to the overriding rule that no candidate is ever questioned for a second time. However, I do not have any very convincing argument that this variant is the best, and there are other possible variants (e.g. restrictions could be placed on the "total flow" of votes transferred in and out of each candidate, or we could stop the negotiation once certain conditions are satisfied).

2 Asset advantages

a1. Asset voting tends to cause more-popular candidates to win, as opposed to simply attempting to get a representative cross section of the electorate without any regard for quality of the candidates as perceived by the voters.

a2. It tends to cause the winners to represent the electorate as a whole – as opposed to zeroing out minority groups.[1]

a3. It encourages peaceful harmony among candidates (since they must negotiate).

a4. In the single-winner special case $W = 1$ it will always elect a true-majority winner (i.e. one with more than 50% of the total votes) if one exists. (But if there is none, then it remains possible that the other candidates can combine forces to defeat the top vote-getter.) More generally, any candidate getting more than $1/(W+1)$ of the votes can guarantee to be among the W winners.

a5. The well known defect of plurality voting in 3-way single-winner elections – the logic that voting for the apparently least popular candidate is a "wasted vote" – seems absent in asset voting, because the third candidate is free to tip the election among the top 2 in any way he likes. Mike Ossipoff called this defect the "favorite betrayal" property – voters in 3-way plurality elections often can feel tactically forced not to vote for their favorite candidate. It appears that in $N = 3$, $W = 1$ asset voting elections, favorite betrayal does not happen.

a6. Asset voting neither assumes nor requires political parties to exist.

a7. It offers voters very high levels of expressivity.

a8. It is monotonic (at least in the pre-negotiation stage) since it is additive. There is no issue of "nontransitive cycles of voter preferences." Giving more of your vote to a candidate can never hurt him.

a9. It is algorithmically near-maximally simple.

a10. No voter needs to do anything more than once (as opposed to some multi-round or "runoff" systems).

a11. It seems immune to manipulation by artificial introduction of "cloned" candidates. That is, if all voters split their votes for a candidate among his clones in some fashion, and if all mutual clones agree to collaborate and "act as one" during the negotiation stage, then the committee output by asset voting will be the same (up to replacements of members by clones) as it was before. This had been a well known defect of plurality voting: introducing a clone of a candidate causes his "vote to be split" often causing both clones to lose an election one otherwise would have won. Some multiwinner schemes have the opposite property – cloning a candidate causes that "team" to be *more* likely to win. It seems best for neither splitting nor teaming to work, i.e. cloning should have little or no effect on the composition of the output committee.

a12. It seems to minimize the probability of near-tied elections (which have a bad habit of plunging the USA into crisis).

STV voting schemes involve a large number (up to N) of "rounds" in which either candidates are eliminated, or are declared to be winners because their vote counts exceed the "Droop quota"[2]. Every round offers the terrifying possibility of a vote near-tie and consequent crisis. In contrast, systems like multiwinner plurality (v14 and v15 of §5) involve only *one* possible tie, and at least in that respect are far less likely to induce a crisis than, i.e. are far superior to, STV.

It is debatable how true a12 really is for asset voting. In some sense asset voting completely ignores issues of who got more votes than whom, and hence ties are irrelevant to it and we have superiority even over multiwinner plurality. But of course, in reality undoubtably very small shifts in certain vote counts could completely alter the nature of the negotiation stage.

3 Possible disadvantages

d1. A "unanimous Condorcet winner" U (whom every voter awards a greater vote to, than to any other candidate) can fail to be among the winners, even in a single-winner election. For example, make every vote be $(.4, .3, .3)$ and then the two lowest-ranked candidates join forces. (However, see advantage a4, and this pathology cannot occur if the candidates themselves feel the same way about U as do the voters.) Personally, I do not regard this as a "disadvantage" – at least not in this example.

d2. Even if 51% of the voters rank some candidate L least-preferred in their votes, he can still be among the winners. An example with $N = 2$ and $W = 1$: Simply make 51% of the voters vote $(.6, .4)$ and 49% vote $(0, 1)$. Again, personally, I do not regard this as a "disadvantage," at least not in this example.

d3. Consider electing a 2-member sailboat-design committee from among 100 candidates. Suppose it is obvious that this committee, in order to accompish anything, requires expertise

[1] More precisely: if there are disjoint kinds of people who always vote for their own kind of candidate, and candidates of the same kinds agree to collaborate during the negotiation stage, then if there are a sufficiently large number of voters and candidates so that rounding-to-integer effects are comparatively negligible, then the committee output by asset voting will have the same composition as the electorate. More precisely and strongly: type-t people will be able to assure election of $W_t \ge \lfloor V_t(W+1-\epsilon)/V \rfloor$ type-t winners, where N_t is the number of candidates of type t from among the N total candidates, V_t is the number of voters of type t from among the V total voters, and W is the total number of winners. (This holds provided $N_t \ge \lfloor V_t(W+1-\epsilon)/V \rfloor$; we use ϵ to denote an arbitrarily small positive real.) The proof is an immediate consequence of a4.

[2] After Henry Richmond Droop (1831-1884).

in *both* hull design and sail design. Of the 100 candidates, 10 have hull expertise and a disjoint 10 have sail expertise.

This kind of scenario – where there are interrelations and interdependencies among the committee members, so that the "net worth" of a committee is *not* described solely by the sum of the net worths of its members – often arises. Then a more accurate approximation would also involve a sum over the net worths of *pairs* of committee members.

However, asset voting does not allow voters to describe such interrelations in their votes. Crudely speaking, asset votes are "linear" whereas "quadratic terms" would be needed to describe this sort of effect. Almost all the voting systems we survey in §5 also are subject to this same criticism.

Thus, in the sailboat example, it seems reasonably likely (probability$\gtrsim$1/4?) that any of these voting systems would elect a completely disfunctional committee!

On the other hand, any system which *did* demand that voters specify information about candidate-pairs would run into the obstacle that $\binom{N}{2}$ pairs exist – corresponding to asking a lot from each voter! For example $\binom{100}{2} = 4950$, and asking a human voter for 4950 pieces of information – even 4950 single bits – seems unreasonable. (And, even if this were regarded as reasonable, then consider the fact that information about candidate-*triples* could also be important...)

Although it is subject to this possibly-severe disadvantage, in my opinion asset voting still is better than most competing systems (§5-6) in this respect. That is because there is a decent chance, in the later negotiation stages, that the negotiators will be able to recognize the disfunctionality threat and correct for it. In contrast, in any purely mechanical system such as STV, there is *zero* possibility for such a correction.

d4. If the "candidates" are actually *not* human beings, but rather some more abstract entities, then our "negotiation" stage becomes infeasible.

An example where this happens is inside "genetic optimization algorithms." Inside each "generation" in such algorithms, it is desirable to choose a "winning" subset of the available entities, and "mate" them in a subset of all possible ways to produce a subset of all possible "children," which then continue on to the next generation. The ultimate goal is, after a large number of generations, to produce an entity with very high "fitness" via "Darwinian evolution." The design of genetic algorithms is currently a black art, and when designing these selection-and-mating procedures it is desirable both to encourage high fitness, and also to preserve "genetic diversity," which are vague and conflicting goals.

d5. Many people feel uncomfortable about having any inter-candidate negotiation (whose results could be unpredictable) at all. Some feel an election ought to be completely predictable from the votes *alone*.

Others worry that the negotiation stage could sometimes grant too much power to tiny sliver groups. For (the most extreme) example, consider the 2000 USA presidential (single winner) election, in which, roughly, A.Gore got 49%, G.W.Bush got 49%, and R.Nader got 2%, of the vote. Suppose these same vote counts had occurred in an asset voting election. Nader would then have the power to choose the winner. What would then happen?

Assume idealistically that Nader, Gore, and Bush were motivated solely by their desire to do (their own notions of) good for the USA. Then it would follow that Gore and Bush (each believing he would do the USA more good as president than the other) would be willing to offer Nader concessions (which, in their view, would lessen total goodness) *up to, but not exceeding* the amount by which they felt they could do more good than the other. So if Bush felt he could do the USA 157 units of good, whereas Gore[3] would provide only 142, then Bush (considering $157 - 142 = 15$) would be willing to offer Nader concessions decreasing his (Bush's) view of the total good by 14 units, but would not be willing to offer 16. In this case, Nader's power actually would *not* be hugely larger than his voter support would seem to justify (and in many repeats of this election in alternate histories, Nader often would be unable to swing the election and hence on average, he would have even less).

But now assume – more cynically and less idealistically – that at least one among Bush and Gore is willing to go to any lengths to attain the presidency. In that case, Nader would have tremendous power as kingmaker and would be able to determine nearly the entire agenda for the president's term, award himself a decent fraction of the federal treasury as a payoff, et cetera. Many people find that a disturbing, and too-likely, possibility.

However, suppose asset voting is only employed for *multi*winner elections (as was its original design intent). Then somehow the potential for harm seems lower. The analogue of "Nader" could now swing, say, the outcome between the election of 3 Republicans and 3 Democrats, instead of the election of 4 and 2. These 6 winners in turn would (in most democracies) only constitute a small part (the representatives from one 6-winner district) of a much vaster parliamentary body. In many districts "Nader" would be unable to swing anything, so averaged over all districts the effect of all the "Naders" would be fairly small, as it should be.

Furthermore, this kind of effect could be viewed as an *advantage* rather than a disadvantage. The "Naders," with 2% of the vote, would *deserve* 2% of the power. But in non-asset voting schemes with 6-winner districts, since 2% is much smaller than 1/7, they would be "zeroed out" – able to obtain neither seats nor power. Another scenario would be a 6-winner district with (if we temporarily ignore its third party voters) 44% Democrats and 56% Republicans. Since $2/6 \approx 33.3\%$ while $3/6 = 50\%$, the election result that would happen in a non-asset system (2 or 3 Democrats and 4 or 2 Republicans) would be forced, by rounding-to-integer effects, to be somewhat unfair. If 3 Republicans were elected, then in a sense 6% of the populace would be being unfairly "ignored." Asset voting can ameliorate both of these kinds of unfairness by causing the "unfairly ignored leftover" votes to represent extra power at the negotiating table, i.e. to amount to something. In this sense asset voting has the potential to achieve *more* fairness than *any* conventional (negotiation-free) voting scheme, by *not* ignoring the votes left over beyond integer-roundoff threshholds.

[3] After himself conceding whatever he would, in Bush's estimation, concede to Nader.

4 The 2002 French Runoff election

The French conduct presidential elections by plurality voting with (if necessary) a second "runoff" election between the top two finishers. This makes it possible for these top two to negotiate with smaller parties between election rounds in an effort to gain their support. Some defects of the French runoff scheme were famously exhibited in the 2002 election.

The two major contenders, respectively leading left- and right-wing coalitions, were Lionel Jospin and Jacques Chirac. However, the left-wingers fielded three additional candidates from minor parties in the coalition, and part of the left-wing vote went to candidates from far-left parties, presumably to protest Jospin's too-centrist policies. The third important candidate was Jean-Marie Le Pen. His platform called for banning immigration, he once described the Holocaust and gas chambers as "a detail of history," and he was known to have practiced torture while stationed in Algeria (he sued the Newspapers accusing him of that for libel, but lost the case). In the first plurality round,

Chirac got 19.88% of the vote; added to Françõis Bayrou's 6.84% (since he was in the same coalition) this is 26.72%. Jospin got 16.18% of the vote; added to other candidates in the same coalition Noël Mamére (5.25%), Robert Hue (3.37%), and Christiane Taubira (2.32%) we get 27.12%.

While we cannot know for sure that voters for minor candidates of each coalition would have voted for that coalition's major candidate, these sums suggest that Jospin had a slight edge over Chirac. But Le Pen obtained 16.86% and thus went to the second round of election, against Chirac, who beat him by a huge 82.21% margin.[4]

So it is plausible in this case that Jospin "should" have won and the French system "robbed" him; and it also is plausible that asset voting would have "correctly" elected Jospin.

5 Theoretical comparison with 18 other multiwinner schemes

Subjectively: I think asset voting is superior to all previously proposed multiwinner election schemes that I know of. This section will back that claim up by analysing 18 different schemes – including the methods presently used in every democratic country, as well as other schemes I either invented myself, or extracted from the literature.

Exception: Some STV schemes seem to be the best previous multiwinner election methods available and in fact may be superior to asset voting. Therefore, we will discuss them seperately in §6.

Completely objectively: I do not know of any previously-proposed scheme which features advantages a1-a12 simultaneously.

However, there are numerous caveats:

1. My knowledge of other schemes is unlikely to be all-encompassing.
2. There undoubtably are many possible voting schemes nobody has yet invented.
3. The question of which multiwinner scheme is the "best" is extremely vague. I do not even know how to formulate the problem.
4. There are probably an immense number of possible bizarre behaviors of such voting schemes, most of which have never occurred to anyone.

Consequently any claim that asset voting is "best possible" is presently very conjectural and speculative.

We now briefly discuss 18 possible alternative voting schemes, and for each we shall point out a disadvantage.

v1. Single winner pseudo-election. I had previously [25] pointed out the advantages of *range voting* in single-winner elections. That is, each voter in an N-candidate election submits a real N-vector, each of whose coordinates lies in the real interval $[0, 1]$, as his vote. The vote vectors are summed and the winner corresponds to the maximum coordinate in the summed vector.

So, we could regard a *multiwinner* election with N candidates and W winners as a single-winner election with $\binom{N}{W}$ pseudo-candidates (the possible committees), and use range voting to determine that winner! This scheme would have the advantage that, *if* each voter were honest and provided the true-utility of that committee (for him) for every possible committee as his vote, then the winning committee would be the genuinely best one (utility-sum-maximizing) for all society.

Unfortunately, hopes for such voter honesty are unrealistic. Even if they were, this idea would require each vote to include a ridiculously huge amount of information, namely $\binom{N}{W}$ real numbers. For example $\binom{50}{25} = 126{,}410{,}606{,}437{,}752$.

Also, this scheme would lack advantage a2: Suppose $N = 2W$ and W of the candidates are "Republicans" and the other W "Democrats." If 51% of the voters award the unique all-Republican committee the maximum vote, then it will win regardless of what the other 49% of the voters do. The Democrats would be "zeroed out," which seems unrepresentative.

v2. Single winner pseudo-election with functions as votes. To get over our "tremendous information" objection to v1, we could ask that each voter v provide as his vote, *not* $\binom{N}{W}$ numbers, one rating each possible committee, but instead a *function* F_v mapping committees to numbers.

(Note that by allowing these functions to include quadratic terms, the potential exists to overcome disadvantage d3.)

The committee C maximizing the sum $\sum_v F_v(C)$ of all the functions would win. Disadvantages: First of all, it may not be feasible to demand this kind of vote from typical voters. Second, it is NP- and APX-complete even to maximize *quadratic* functions over the N-cube, so that such an election would be algorithmically infeasibly difficult. (The NP-completeness proof is an easy reduction from MAXIMUM 2-SATISFIABILITY [11].) Certain specially restricted forms of the functions might lead to algorithmic feasibility, though,[5] and some examples will be discussed later.

[4] Arguably supporters of Le Pen's former deputy and extremist rival Bruno Mégret (2.4%) would also have supported Le Pen.

[5] E.g. if the functions are linear.

v3. Multiwinner Condorcet least-reversal voting. Each voter supplies a permutation of $\{1, 2, 3, \ldots, N\}$ – a preference ordering of the N candidates – as his vote. We construct an N-vertex directed graph with a directed arc pq if candidate q is preferred to candidate p by the voters (with all candidates other than p and q are erased from all votes, then regarding this arc as a 2-candidate 1-winner sub-election). Each arc is labeled with the numerical margin of victory for that 2-candidate sub-election. Now we *reverse* the direction of some subset of the arcs. Namely, the arc-subset with minimum possible total sum (of the numerical margins) is chosen subject to the constraint that it will cause the digraph to contain a W-element subset of "winner" vertices, such that all the arcs between a winner and a nonwinner, point to the winner.

V3 is *not* algorithmically simple (i.e. disobeys a9): In fact finding the required min-weight arc set to carry out the election is an NP-complete task! (With $W = 1$ it is always linear time, though.) The NP-completeness proof is an easy reduction from GRAPH PARTITIONING [11]. However, for elections with $N \le 30$ candidates, this is no obstacle since it is feasible to consider all $\binom{N}{W}$ node-subsets exhaustively. Furthermore, there is a different Condorcet-like method, due to N.Tideman [31] which has the advantage that determining the W winners *is* in P:

v4. Multiwinner Tideman. Each voter supplies a preference ordering of the N candidates as his vote. In the Tideman system, you pick the $A > B$ candidate-comparison with the largest margin and "lock it in." Then you pick the next largest one available ("available" means: not already used and not creating a cycle in the directed graph of candidate-comparisons) and continue on. This ultimately creates an ordering of the candidates. The topmost W in the ordering win. The Tideman system is equivalent to least-reversal Condorcet variant if there are $N \le 3$ candidates; but if $N \ge 4$ they can differ.

More seriously, both v3 and v4 can yield a very nonrepresentative winner subset, disobeying a2. Suppose $N = 2W$ and W of the candidates are Republicans and the other W Democrats. If 51% of the voters voted a "Republican straight ticket" (preferring every Republican to every Democrat, but the preferences *within* the Republicans are random and the number of voters is very large) then the W winners will be 100% Republican no matter what the other 49% of the voters do.

Tideman's system is supposed to be immune to clones. That is true in the sense that, with honest voters, it will yield the same rankings of all clones in the output-ordering as the original uncloned candidate. However it is false in the (for us, more important) sense that the composition of the winner-committee will be altered!

v5. Multiwinner Range Voting. Each voter generates N numbers in the real interval $[0, 1]$ as his vote. The vote N-vectors are summed and the greatest W components in the sum correspond to the winners.

But again this scheme would zero out Democrat candidates. Again suppose $N = 2W$ and W of the candidates are Republicans and the other W Democrats. If 51% of the voters voted a "Republican straight ticket" $(1, 1, 1, \ldots, 1, 0, 0, 0, \ldots, 0)$ then again the W winners will be 100% Republican no matter what the other 49% of the voters do.

v6. Multiwinner Coombs-STV Voting. Similar to Hare-STV (discussed in §6), except the candidate eliminated each round is the one with the *most* last-place votes.

This is a very bad system because strategic voters will rank their least-liked among the favorites artificially "last." This will cause all the favorites to be eliminated in early rounds, causing the winners to consist entirely of unknown "dark horse" candidates.

Another flaw in Coombs is its vulnerability to cloning. If candidate A is preferred by a 75-25 margin over B in a single-winner election, then upon replacing B with 4 clones B_1, B_2, B_3, B_4, the 25% of the population that hates A will eliminate him (since the hates-B vote is split 4 ways, 20% to each B_i) in the first round.

v7. Multiwinner Borda Voting. (Essentially the system used in Norwegian internal party elections.) Each voter supplies a preference ordering of the N candidates as his vote. Then each time a candidate gets a Kth place vote he is awarded a score of $N - K + 1$ and the W candidates with the greatest score-sums win.

This again fails the straight-ticket Republican versus Democrat test. It also has less voter-expressivity than asset voting and in the single-winner case experimentally is worse than range voting [25]. Also, "teaming" occurs: parties tend to be rewarded by Borda for running many identical candidates.

v8. Multiwinner Nanson [24] Voting. Voters provide preference orderings as their votes. We proceed in rounds. In each round the candidate with the smallest Borda score is eliminated (both from the election, and from all preferrence orderings). This continues until only W remain; they are the winners.

This scheme seems vulnerable to manipulation by candidate cloning, although its effect now seems *beneficial* to a candidate ("teaming"), but this is still a Bad Thing. In a straight-ticket voting 51-Republican 49-Democat scenario, the Democrats will be entirely eliminated.

v9. Voting by convex programming. Each voter k supplies a real-valued differentiable concave-$\cap$ function F_k of N real variables (each in $[0, 1]$) as his vote. The sum $F = \sum_k F_k$ of all such functions is also real-valued differentiable concave-$\cap$. By convex programming [18][30] it is algorithmically feasible (i.e. in P [11]) to find the maximum of any such function on the N-cube $[0, 1]^N$ intersect the hyperplane $\sum_{n=1}^{N} x_n = W$, to high accuracy. The N-vector representing the location of this maximum is regarded as describing the winning committee.

Unfortunately, this committee may have "fractional members," i.e. the vector may not have entries that are 0 or 1 only, but instead it may contain fractional entries. In general no algorithmically efficient way is known to find the best rounding off to pure-{0,1} values. (It is relevant here that INTEGER PROGRAMMING and the problem of finding the maximum of a concave-$\cap$ quadratic function on the discrete N-cube $\{0, 1\}^N$ are both NP-complete [11]. Indeed the latter is APX-complete.)

Surprisingly, fractional members might actually sometimes be acceptable. We simply choose person X to be on the committee (or to participate in some particular committee activity) with probability equal to his fractional membership. This would be a nondeterministic voting scheme in which chance played a very substantial role, but it conceivably would be better on average than any deterministic method. I'm rather dubious of this whole idea, but have no convincing argument against it.

v10. Voting by convex programming (more enjoyable special case). Each voter k supplies a real N-vector $\vec{v}_k$ as his vote. These N-vectors all are required to obey the condition that the sum of their greatest W entries is 1 and the sum of their least W entries is 0 (a normalization which could be assured automatically by a linear transformation, if the voter himself did not want to bother). The scheme will involve a fixed concave-$\cap$ monotonically increasing self-map g of $[0,1]$, for example $g(u) = u^e$ for some fixed constant e with $0 < e < 1$. The winning committee is the one whose Boolean membership vector $\vec{x}$ maximizes

$$F(\vec{x}) = \sum_{\text{Voters } k} g(\vec{v}_k \cdot \vec{x}). \tag{1}$$

This would still be equivalent to range voting in the $W = 1$ case, albeit the equivalence mapping is a little involved.

By choice of g (or in the case $g(u) = u^e$, by choice of e) this scheme could be made, in the Republican-Democrat straight-ticket 51-49 scenario, to elect some nonzero fraction of Democrats, and that fraction indeed would be adjustable by varying e.

Again it is algorithmically feasible (i.e. in P [11]) to compute the $\vec{x}$ with components in $[0,1]$ (and with $\sum_{j=1}^{N} x_j = W$) which maximizes $F(\vec{x})$, since this is a "convex programming problem" [30]. Unfortunately, again the answer could be a committee with "fractional members"! To enforce genuine 0,1 membership requires solving a "convex $\{0,1\}$ programming" problem, but that might be much harder computationally, requiring an exhaustive backtrack search among all the $\binom{N}{W}$ possible committees, which if N is large enough ($N = 100$?) might become computationally infeasible.

v11. Range voting with negotiation. Each voter's vote is an N-tuple of reals in $[0,1]$. These vote-vectors are added to get each of the N "vote-totals" and then inter-candidate "negotiations" proceed as in the scheme of the present paper to determine the W winners.

Disadvantage: Cloning a candidate many times will cause more of him to get elected.

v12. Asset voting without negotiation. Each voter's vote is an N-tuple of nonnegative reals summing to 1. These vote-vectors are summed and the W greatest coordinates in the sum-vector correspond to the W winners.

Fails the Republican-Democrat 51-49 straight-ticket voting test. Cloning causes damaging vote-splitting effects.

v13. Party-based plans. (Used in many countries including Netherlands and Israel.) Voters vote for a political party, not for candidates directly. Then the parties somehow get together to determine the winners (there are many possible ways for them to proceed at this point).

Disadvantage: This requires the existence of political parties. It causes voters to have no direct input (or at best only a subset of them have input) about the question of individual preferences among the candidates from any single party. The system is particularly outrageously undemocratic as implemented in Israel, where the entire country (there is no districting) votes simply for a party (a vote is just the name of one party) and the party itself then decides who shall get whatever number of seats it then has[6].

v14. Multiwinner "non transferable" plurality voting. (Used in Japan.) Each voter names a single candidate as his vote. The W candidates with the most votes win.

This again fails the straight-ticket Republican versus Democrat test. It also is subject to all the well known defects of plurality voting, including cloning and favorite-betrayal.

v15. Multiwinner multivote plurality voting. (Variants used in several democracies and in Philadelphia City Council elections.) Each voter names K candidates as his vote (for some constant K). The W candidates with the most votes win.

This again fails the straight-ticket Republican versus Democrat test and the cloning test.

v16. "Cumulative" vote. (Formerly used in Illinois state house elections and in British school board elections in the late 1800s.) Like v15 except that repeating a name is allowed, i.e. the K names are a multiset which is not required to be a set.

This again fails the straight-ticket Republican versus Democrat test and the cloning test. It is, however, groping toward the scheme we are advocating in this paper; in the limit $K \to \infty$ this is essentially the same as v12.

v17. Hamming distance-sum minimization.[7] Each voter names a subset of the candidates as his vote. (We could, if desired, demand this subset have cardinality W.) The specification of this subset could be regarded as the '1' bits in an N-bit binary word. The winners are a W-element subset (viewed again as a binary word) with minimum total summed (Hamming distance)^e to all the voters. (Here e is some fixed real in $[1,\infty]$.) In the limit $e \to \infty$ we get v18. In the case $e = 1$ this is just multiwinner approval voting, which we have criticized in v5.

If $1 < e < \infty$ then it may not be algorithmically feasible (i.e. may not be in P) to find the min-cost winner set. However, finding the N-vector with elements in $[0,1]$ which minimizes the sum of the eth powers of the *Euclidean* distances to the vote-vectors *is* algorithmically feasible (convex programming). Allowing vote-vectors to be real rather than Boolean seems superior anyway for the purpose of giving more expressivity to the voters. Unfortunately the result is a committee possibly including "fractional members" as in v8, and finding the best N-vector with Boolean elements $\{0,1\}$ may be algorithmically hard.

[6] Israeli Prime Minister Ariel Sharon sacked his deputy infrastructure minister Naomi Blumenthal in December 2002 after she refused to answer questions from police about vote-buying. The Likud party selects its candidates for parliament through a vote of its central committee. (Note: *not* the voters themselves.) Blumenthal won a prominent slot on the Likud Knesset candidate list and was charged with paying for rooms at a posh Tel Aviv hotel in exchange for votes by central committee members.

[7] The "Hamming distance" between two equal-length binary words is the number of bit-positions in which they differ. For example the Hamming distance between 11111011011 and 11**0**110**0**1**10**1 is 4.

In the straight-ticket Republican-Democrat 51-49 scenario, v17 unfortunately would elect a 100% Republican committee.

v18. Voting via minimum enclosing Hamming-sphere [5]. Same thing as v17, but the winners are the W-element subset (or binary word) with minimum maximum Hamming distance to all the voters.

Disadvantages: often produces a non-unique result; algorithmically may not be in P; ignores duplicated votes, no matter how many duplications there are; in the limit of a large number of voters (exceeding 2^N) this method tends to become completely equivalent to "pick a random result" because the min-enclosing sphere simply becomes the entirety of Hamming space.

If we optionally allow votes to be *real* N-vectors with components in $[0,1]$ and instead find the minimum-radius enclosing *Euclidean* sphere, then this is algorithmically feasible [12][13][14][32]. but as in v8, the solution (the vector that is the sphere center) may be a committee with fractional membership, i.e. be a vector whose entries lie in $[0,1]$ but not necessarily $\{0,1\}$. Again, requiring true 0,1 membership may be algorithmically difficult. This Euclideanized scheme still would ignore duplicated votes.

Remarks. Various other voting systems (most of them hybrids of subsets of the ideas above) are discussed on the ERS web site [9], the appendix of [23], the books [21][22], and throughout the excellent special issue *J.Economic Perspectives* 9,1 (1995). For information about which schemes are used in which governments see [6][8][21][23][29].

In the above discussion I have sometimes taken for granted some understanding of computational complexity theory, in particular the theory of P and NP and sometimes APX. For P and NP see [11] and for APX see [2].[8] We follow the usual rule of thumb that algorithms in P can be regarded as "feasible" whereas NP- and APX-complete tasks should be regarded as algorithmically infeasible and hence not recommendable for application inside voting systems. *Linear time* algorithms are the creme de la creme, the best algorithms in P. Asset voting (assuming the input consists of fixed-precision fixed-point reals in a suitable order) is in that class.[9]

6 Comparison with STV methods

The two crucial ideas behind STV (Single transferable Vote) methods were first invented by Thomas Wright Hill (England 1763-1851; invented idea of vote "transfer") and Carl G. Andrae (Denmark 1812-1893; invented "quota" idea). An independent discovery was made by Thomas Hare (England 1806-1891) and described in his book *Treatise on the election of representatives, Parliamentary and Municipal* (Longman, London 1859). Later revised editions came out in 1861, 1865, and 1873, with the 1865 edition including a further refinement, the introduction of "elimination." An important final refinement was the "Droop quota" created in 1868 by Henry Droop and described in an 1881 paper [8]. The resulting system is clearly described, with a worked example, in chapter 7 of [22]. The best description I know of the conceptual development of STV is Tideman's paper [28], which is also the only reference I've seen that includes the "proportionality theorem" statement and a proof.

An STV-advocacy group, the Proportional Representation Society, formed in Britain in 1884. During the next 120 years, they apparently had no success at causing Britain to switch to STV, but did have some impact on Ireland, Australia, New Zealand, and Malta, all of whose governments now employ STV to some degree. The PR Society now call themselves the Electoral Reform Society [9] and they now advocate a voting procedure called "ERS97" which is similar to our pseudocode below, but with additional refinements.

Here is a pseudocode description of the Hare/Droop STV procedure. Let there be N candidates, from whom V voters are to choose W winners ($0 < W < N$, $0 < V$).

procedure STV-election

Obtain from each voter a preference ordering (permutation) of the N candidates;
Associate each vote with a real "weight" w with $0 \le w \le 1$, where initially all weights are 1;
Compute the "Droop Quota" $Q = \lfloor V/(W+1) \rfloor + 1$;
loop
  repeat
    for $c = 1$ to N **do**
      Compute F_c, the sum, over all votes ranking candidate c first, of that vote's weight;
    end for
    $g = \text{argmax}\, F_c$;
    $\{g$ is the "good" canddt with the most 1st-place votes$\}$
    if $F_g \ge Q$ **then**
      Multiply the weight of each vote which ranks g first, by $(F_g - Q)/F_g$;
      Declare g to be a "winner" and eliminate g from all preference orderings;
    end if
    exitwhen W canddts have been declared winners;
  until $F_g < Q$
  $b = \text{argmin}\, F_c$;
  $\{b$ is the "bad" canddt with fewest 1st-place votes$\}$
  Declare b to be a "loser" and eliminate b from all preference orderings;
end loop

This is a fairly complicated procedure. Many variants of it, both less and more complicated, also exist. One of the best seems to be Meek's weighting scheme [19][10], but it is much more complicated, e.g. involving a nonlinear multivariable iteration to convergence.

The most important theorem about `STV-election` is the **proportionality theorem** [28]: This postulates that the voters and candidates consist of several disjoint types of people, and each voter of a given type always prefers each candidate of his same type, above every candidate of any other type. Let the

[8] Our text has sometimes stated minor theorems concerning the NP- and APX-completeness, or linear-time nature, of various computational tasks – but not proved them. That is because all of these theorems are of the "undergraduate textbook exercise" level. We admit, though, that to anybody completely unfamiliar with computational complexity theory, they could seem mysterious.

[9] Hint: employ an "adder tree." This will lead to a linear-time algorithm for adding N numbers, each B bits long, on a 2-tape Turing machine.

[10] According to [29], Meek-STV may now be used in some New Zealand elections.

number of voters of type t be V_t, the number of candidates of type t be N_t, and the number of winners of type t be W_t. Then: $W_t \geq \lfloor V_t/Q \rfloor$ if $N_t \geq \lfloor V_t/Q \rfloor$.

I have actually stated a stronger version of this theorem than was previously available [28]. Nevertheless, due to the rounding-to-integer effect inherent in the Droop quota formula in line 3, this still is not as strong a proportionality theorem as we showed for asset voting in footnote 1; because asset votes are reals and not integers, better proportionality is achievable.

Proof: Realize that any candidate with more than Q of the first-rank votes is assured of election, and then regard the first rank votes as being continually "transferred" between the candidates, with any candidate exceeding the Droop quota Q transferring all his excess votes above Q. The total number of available first-rank votes is always V and of first-rank votes for type-t candidates is V_t. If candidates always transfer votes to their own kind of candidates whenever they are still available, evidently the number of candidates of type t who will acquire $\geq Q$ votes, is $\lfloor V_t/Q \rfloor$. Q.E.D.

Warning: There are various **simpler** STV procedures which, however, don't obey the theorem.

This theorem depends for its validity on the Droop quota Q (line 3) and its use in determining "early winners" (lines 11-14) and in reweighting (line 12). It would have been possible to devise a much simpler **elimination-only** variant STV scheme in which lines 5 and 9-16 (except for line 15) were omitted (and in fact this variant is equivalent to our full algorithm, and also to Meek and ERS97, in the single winner case $W = 1$). However, that variant would not have satisfied the proportionality theorem because of the following counterexample.

Let $N = 2W$, let there be two kinds of people ("Democrats" and "Republicans"), and let 49.99% of the voters be Democrat and 50.01% Republican. Let there be W Republican and W Democratic candidates, with all the Republicans being indistinguishable "clones" (whom the voters order randomly) but with one special Democrat being more attractive to each of the Democratic voters, than any of the other $W-1$ Democratic candidates. Then, no matter how large W is, elimination-only STV will elect a committee consisting of the one special Democrat and $W-1$ Republicans (very unrepresentative). That is because the unspecial $W-1$ Democrats will be eliminated since each has zero first rank votes.

In contrast, the full `STV-election` procedure would have elected the special democrat immediately (exceeds quota), and then eliminations would have roughly alternated between declaring Republican and Democrat losers, until at last a committee with 50-50 composition was attained.

It would instead have been possible to simplify `STV-election` to *get rid* of the *elimination* steps in lines 17-19. However, the resulting procedure could loop forever with no candidate ever reaching the quota necessary to declare him a winner! (This would in fact usually happen if all vote orderings were random.) To fix that problem we could (further) eliminate the quota entirely (kill lines 3, 9, 12, and 16) to simply elect the candidate with the most first-rank votes each round. Unfortunately that kind of **simple transfer** procedure also would elect nonrepresentative committees: in the above Republican-Democrat scenario, but without the special Democrat (all Democrats now are randomly-ordered clones too), a 100% Republican committee would be elected.

Now that we understand Hare/Droop STV and its properties, we are in a position to compare it with asset voting. Both `STV-election` and asset voting share advantages a1, a2, a4 (appropriately reworded), a5 (in some sense), a6, a10, and a11, and both share alleged disadvantage d2.

Another shared disadvantage is that neither reduces to range voting in the single-winner case (which is experimentally the best single-winner voting system [25], at least among known conventional systems in which the possibility of "negotiation" is forbidden). This is irrelevant in practice if we are only using these schemes for *multi*winner elections, but it certainly is theoretically bothersome. In 3-candidate 1-winner STV (also Condorcet) elections, it can be tactically wise to vote dishonestly [25][7][16], whereas in a 3-candidate 1-winner range-voting election, the tactically best vote always has $\leq$ relations compatible with (i.e. a valid limiting case of) that voter's true preference ordering. Arguably tactical *asset* voting in 3-candidate 1-winner elections also is honest (which if so would be another advantage for asset voting over STV) but due to the haziness of "negotiation" it is difficult to be sure of this.

Asset voting is superior with respect to a3, a7 (in STV there is no way for voters to express the *intensity* of their preferences), a8 (in STV, even in the single-winner case, your vote ranking some condidate *first* can cause that candidate to *lose* [3][7]), a9 (asset voting is clearly simpler, e.g. requiring only addition without multiplications or divisions), and a12.

Asset voting is inferior with respect to disadvantages d3 and d4 (because it requires an intercandidate negotiation) and with respect to alleged disadvantage d1.

Because STV is worse with respect to a3, a7, a8, a9, and a12, our claim that "no known" multiwinner election system has advantages a1-a12 remains technically true. However, in my view, none of these superiorities or inferiorities represents a "knockout punch," so I cannot make any blanket statement that one or the other among Hare/Droop STV (or some of its variants) and asset voting is clearly superior in all applications. (In contrast, I *do* view the voting systems v1-v18 in §5 as having suffered a knockout punch.)

Despite that, I will say that asset voting is so *tremendously simpler* than `STV-election` that, in any moderately large election run without the benefit of computers, it almost certainly is to be preferred. In practice STV is even more complicated than we have sketched, because of the needs (1) to allow voters to express equalities in their preference orderings, (2) to do something about voters who rank-order some but not all of the candidates, and (3) possibly to break ties. In contrast, in asset voting issue 3 is nonexistent, issue 1 is trivial, and issue 2 is easily dealt with because we can just allow the voter to say "...and award X votes to each of the remaining candidates" for some voter-chosen value X.

Furthermore, in asset elections held at several voting locations simultaneously, each location can simply transmit its subtotal, once, to a central agency. In STV elections, on the contrary, the required communications could be *far* more voluminous (e.g. every single vote) or time consuming (e.g. many rounds of 2-way communication, with possible need to restart every-thing each time any error is found or a new, as-yet-uncounted, vote comes along).[11]

7 Experimental evaluation in the single winner case

I do not know how to experimentally assess asset voting in the *multi*winner case, because I do not understand the societal (or personal) benefit of electing any particular committee. (If these were known, then it would be possible to construct a mathematical theory of a multiplayer game with known payoffs, etc.)

However, we *do* have a pretty good idea how to compare *single*-winner voting systems [25]: we construct artificial voters and candidates. By some randomized method, we create a private personal utility for the event "candidate C is elected" from the viewpoint of each voter V. We then run an artificial election using some voting method M. (The voters can be made "honest" or "rational-strategic" and/or suffering from "ignorance" – we shall regard those choices all as embraced within the definition of M.) The difference between the society-wide (i.e. summed over voters) utility of the election winner, versus the society-wide utility of the hypothetical maximum-utility winner (with this difference averaged over a tremendous number of such Monte Carlo experimental elections) is the *Bayesian regret* of voting method M. Voting methods with smaller regret are, of course, to be preferred. This is a quantitative measure of how good a single-winner election method is.

In these experiments, it is possible to vary the number of voters and candidates, and to vary the randomized private-utility generator. This in general will cause the Bayesian regret values to change. Thus it always is necessary to compare voting methods when these parameters are the *same*, and it is of interest to see what happens to the comparison when they change.

My previous [25] contained the world's largest such experimental study of single-winner voting systems, and came to the remarkably clear conclusion that *range voting* was superior to all other voting systems tried.[12]

We shall now describe a new and smaller such study, redesigned to allow asset voting to participate. **The experimental setup:**

Each candidate and each voter is regarded as a point on the surface of a unit-radius D-dimensional ball ($D = 1, 2, 3$ were tried), i.e., a unit-norm D-vector. The *private utility* of a candidate, as viewed by a voter, is the *inner product* of the voter's and candidate's vectors.[13] These utilities all therefore lie in the real interval $[-1, 1]$. The candidates and voters are disjoint, i.e. candidates are not allowed to vote. (We shall regard the voters as male and the candidates as female to enhance clarity in what follows.)

All voters and candidates are chosen independently from a *single* common distribution on the D-ball's surface, which we now describe. Fix a parameter μ called the "number of clumps." (Values $\mu = 1, 2, 3, 4, 5, \infty$ were tried.) Fix μ different random unit vectors (picked from the uniform distribution on the D-sphere) called the "clump centers." The method for finding a newborn voter's characteristic D-vector is then to pick a random-uniform vector on the D-sphere, and then test whether it has inner product ≥ 0.9 with any clump-center[14]. If so, the vector is accepted (otherwise we retry). The case $\mu = \infty$ corresponds to the case of uniform distribution on the sphere; but when μ is finite we get a situation where voter and candidate opinions tend to fall into "clumps."

We studied **these 10 single-winner voting methods:**

1. Pick a random winner from among the candidates (completely ignoring all votes). This was included purely to provide a "yardstick" allowing the reader to get a good idea how bad the other methods are. (Not surprisingly, this method always yielded larger regret values than all the others.)
2. Strategic plurality voting. Plurality voting is used. Each voter picks the candidate, from among the first two ("the two most favored in pre-election polls") with the (in his view) greatest utility.[15]
3. Honest plurality voting. Each voter picks randomly among those candidates having (in his view) maximum utility.
4. Strategic approval voting (which is the same thing as strategic range voting [25]). Let $j \geq 2$ be the smallest number such that the first J candidates do not all have the same perceived utility. The voter votes 1 and 0 for them (the higher utility ones get the 1). Then, he goes through the remaining $N - J$ candidates in order. For each, if that candidate's utility exceeds the average of the preceding candidate's utilities, she gets a '1' vote, otherwise a '0' vote.
5. Honest range voting. The voter uses his private utilities for the candidates as his vote, *except* that first he rescales these utilities via a 1-dimensional linear transformation so that the (in his view) best candidate gets a vote of 1 and the worst 0.
6. No-negotiation asset voting. The voter uses his private utilities for the candidates as his vote, *except* that first he rescales these utilities via a 1-dimensional linear

[11] In a future paper [27], I intend to describe a new multiwinner voting system called "reweighted range voting" that attempts to combine the advantages of both STV and Range Voting. This new system reduces naturally to range voting in the single-winner case, and grants voters the same amount of expressivity as in range voting (i.e. real vectors rather than preference orderings, i.e. more expressivity than in ordinary STV). It is monotonic (unlike ordinary STV), and it does not involve any negotiation (unlike asset voting). Unfortunately, although it is simpler than `STV-election`, this new scheme still is considerably more complicated both to describe and use than asset voting, and again in a multilocation election the communication requirements can be very onerous.

[12] This was true no matter whether honest, strategic, or ignorant voters were used, and no matter what the number of candidates and voters, and no matter which of numerous utility generators were employed.

[13] Because $|\vec{x} - \vec{y}|^2 = |\vec{x}|^2 + |\vec{y}|^2 - 2\vec{x} \cdot \vec{y}$, if both vectors $\vec{x}$ and $\vec{y}$ are of unit norm this is just $= 2 - 2\vec{x} \cdot \vec{y}$. This allows satiating the desires of people who prefer thinking about distances instead of inner products.

[14] Thus these clumps are spherical caps with angular diameter $2 \arccos(0.9) \approx 51.68°$; two clump-caps can overlap; and the distribution is uniform within the union of all such caps.

[15] If both have equal utility then he votes for the first candidate with maximum utility.

transformation so that the (in his view) worst candidate gets a vote of 0 and the votes for all candidates sum to 1. The candidate with the most votes wins.

7. Asset voting with plurality-like voting. The voter votes 1 for the (in his view) best candidate, and 0 for all the others. (Probably a close approximation to what real voters, in an effort to be *strategic*, will actually do.) The candidates negotiate as follows: the candidate with the currently least votes hands them all to the as-yet-uneliminated candidate whose utility (from her point of view) is maximal, and then drops out. This continues until only one candidate (the winner) remains.
8. "honest" asset voting, i.e. with utilities as votes: Same voting as in method 6; same negotiation as in method 7.
9. Honest STV. Votes are honest preference permutations (same order as perceived candidate utilities).
10. Strategic STV. Votes are preference permutations got as follows: Rank the favorite among the first two candiates top, and the other bottom. Then rank the remaining $N-2$ candidates honestly (within the remaining $N-2$ slots).

In all cases, ties were broken randomly in any tied elections.

The results of these experiments (with 7 and 101 voters per election) are **tabulated** at the end of the paper. We only tabulate cases with $N = 3$ and 4 candidates.[16] There then are $72 = 2 \times 3 \times 6 \times 2$ experiments in all (each a comparison of 10 different voting systems), corresponding to 2 possible numbers of voters (7 and 101), 3 possible space-dimensions ($D = 1, 2, 3$), 6 possible clump numbers ($1, 2, 3, 4, 5, \infty$), and 2 possible numbers of candidates ($N = 3, 4$). Hence we tabulate 72×10 regret numbers, each of which represents an average over 500,000 elections for the 101-voter elections, and an average over 2,500,000 elections for the 7-voter elections.

These results are very interesting and make asset voting look both very good and very bad, depending on the circumstances. (In over 80% of the cases, though, it looks good.) There were several surprises.

Is negotiation a good idea? When $D = 1$ (the case where all candidate utilities are Boolean) negotiation was always a good idea – since it always got the regret down to zero!

But when $D \geq 2$ our simple negotiation procedure *failed*,[17] in 83% of the cases, to improve things over utility-based asset voting without negotiation, i.e. method 8 usually had at least as large regret as method 6! In fact, in 4% of the cases including negotiation made utility-based asset voting perform worse than *every* other method (except random-winner)![18] In retrospect, this is not surprising. If voters are honest and are allowed to fully express themselves (votes=utilities) then usually[19] the best thing to do, from the point of view of society-wide utility, is to let them have their way! Negotiation only is needed if the voters instead try to be strategic=dishonest, or if they are not allowed to express themselves fully (because, e.g. they use plurality voting and hence can only express information about *one* of the candidates) in which case damaging strategic phenomena such as "vote splitting" enter the picture. The whole purpose of the negotiation stage is to partially repair that damage.

Hence it is more interesting to consider the effect of negotiation after an honest-plurality-like vote (which is our best guess about what strategic asset voting would be like). Here, we find that negotiation improves[20](or at least does not worsen) Bayesian regret in 93% of the cases. In 82% of the cases[21] this improvement was great enough to cause asset method 7 to yield smaller regret than strategic approval voting (method 4) – making it (in those 82%) apparently the *best* voting method with strategic voters known! This is powerful testimony for the usefulness of negotiation!

An unexpected side effect of this study was: realistic examples of situations where *strategic* plurality voters actually yield better societal results (lower regret) than *honest* ones! This only happened in a statistically significant way in 12 out of our 72 experiments[22] – but it is impressive that it can happen at all. Apparently, honest plurality voters in these cases suffer heavily from "vote splitting," and by agreeing to focus on just two arbitrary (but fixed) "major party" candidates, they get rid of enough vote splitting to more than compensate for the loss of honest information.

Honest **range voting** always was at least as good as honest **STV voting**, and strategic range voting always at least as good as strategic STV voting, in all 72 election scenarios, with the single exception that in the $\mu = 1$, $D = 2$, $N = 3$ case (a distribution where, as we've mentioned, honest range voting does particularly badly) honest STV actually was superior to honest range. (This case is the first time that has ever happened naturally, but since this distribution seems fairly unconnected to reality, range voting remains clearly superior to STV.)

The situations that made method 7 look maximally bad – worse than every other (non-random) method tried! – were the $\mu = \infty$ (no clumping) $D = 2, 3$ cases with 101 voters.

[16]With 2 candidates, we confirmed by computer that all voting methods except random-winner had identical Bayesian regret values, because they are identical voting methods.

[17]There were exactly 12 exceptional cases among these 48 experiments in which negotiation improved over an honest utility-based asset vote. All involved 3 or 4 candidates, $D = 2$ or 3, and 2-4 clumps.

[18]Specifically, in three 101-voter elections with $\mu = 0$, $D = 2, 3$.

[19] No-negotiation utility-based honest asset voting was not as good as honest range voting, although it usually was not bad. Amazingly, in the 4-candidate 7-voter 1-clump $D = 2, 3$ cases, it actually was *better* than honest range voting – the first time any realistic voting method has achieved that in a realistic scenario! (For some reason it appears that the 1-clump cases of our distributions are particularly ill-suited for honest range voting.) The worst performances for negotiation-free utility-based asset voting (in comparison with honest range voting) seem to occur when the candidates all seem nearly identical in the eyes of most voters. In that case, most voters will scale their votes to reduce the importance of candidate differences, causing the election results to be dominated by the few freak voters for whom there is one very good candidate. That introduces a lot of "noise" which range voting is comparatively immune to.

[20]In our 72 experiments there were only 5 cases where negotiation worsened an honest plurality vote: all were clump-free (i.e. fully spherically symmetric) cases with $D = 2, 3$.

[21] In our 72 experiments there were only 13 where method 7 was worse, by a statistically significant margin, than strategic approval voting. Furthermore, it was better than strategic STV voting in all but 3 cases (all of which had $\mu = 0$ – perfect spherical symmetry – and 101 voters).

[22]All were cases with 101 voters.

That presumably is because in these cases perfect spherical-symmetry is approached, i.e. all candidates are identical except for small perturbations due to the fact that the random locations of the 101 voters cannot be *exactly* uniform on the sphere. While the non-asset voting methods tend to be comparatively good at extracting the needed statistical information about the tiny differences between the candidates in this case, the negotiation stage of asset voting introduces huge perturbations which arise almost entirely from "noise." It also does not help that the pathology of footnote 18 simultaneously happens in this case, yielding a "perfect storm." In view of this, any claim of the superiority, or even the competitiveness, of asset voting must rest on the assumption that this kind of situation of exact symmetry, in which all candidates are nearly identical, is rare in practice! Fortunately, that probably is a pretty realistic assumption, and when it is not, then arguably the question of who wins the election does not matter much anyway.

8 Conclusion

Philosophy. The first stage of the present paper's scheme converts votes into a conserved **asset** similar to **money.** Then the second ("negotiation") stage allows the laws of economics to enter the picture. Representation is assured since a voter group of say 15% of the voters will control 15% of this "asset," which will enable them to "buy" 15% of the seats. Further, perceived-as-better candidates will get more of the asset, which is good.

It is for this reason that we call our method "asset voting."[23]

What motivates candidates during the negotiation? There are several possibilities. They could care about their ideas and hence would support the candidate whose views were closest to their own. They could care about power and hence would donate their votes to candidates who agreed to give them favors, or "pork," or cabinet posts, in return. They might be influenced by agreements to act in certain ways ("support my health plan and I'll give you my vote") or they could care about money ("buy my votes") – which is somewhat equivalent to power at least in today's US politics.

Some of the above possibilities are more deplorable than others, but all are approximately good because they are doing roughly what the voters want. Anybody who votes for a candidate C does so because he wants C to get more power. Even if C is doomed to lose because he didn't get enough votes, those votes can still suffice to give him power at the negotiating table. There, C can then try to convert that power into other forms, such as power to choose winners, or into favors, pork, policy changes, or cabinet posts. Either way, your vote is accomplishing something in roughly the direction you want (and if it isn't, that is your fault for foolishly voting for C and misjudging what C was going to do). Similarly, votes for a candidate who *does* have enough to win are not wasted because they too give him more power at the negotiating table, which he can use to *avoid* granting concessions to less-popular candidates, and (further) in variant #2 of asset voting, he can use his excess votes to help his "friends."

How good is it? On paper, based on the list of *theoretical* desiderata in §2, it appears asset voting is superior to all *multi*winner election methods known to me. However, those advantages seem insufficiently large to constitute a clear "knockout punch" over the best STV schemes – and asset voting also suffers some theoretical disadvantages relative to those schemes (see §6). Hence, reasonable judges could well still prefer STV, at least for some applications. *Experimentally*, asset voting often appears to be the best available method (when applicable) for *single*-winner elections. This is assuming strategic voters, and assuming our notion of "good strategy" in asset voting[24] is approximately correct. Specifically, in 82% of the experimental scenarios tried, asset voting had comparable or smaller regret than strategic approval voting (the best previously known method). In 96% of the experimental scenarios strategic asset voting was better than strategic STV voting, and although it was worse in the remaining 4%, it was only worse by 9-46%, i.e. not a *lot* worse since STV also performed quite badly in those cases (namely 2.3-2.6 times worse than strategic approval voting). This single winner data suggests that asset voting might also be preferable to STV in *multi*winner elections. In a small percentage ($\approx 7\%$) of the experimental scenarios, the negotiation stage of asset voting on average causes more damage than it repairs, causing disturbingly poor performance (high regret values), worse than plain plurality voting. Perhaps this 7% is trying to tell us something – it may mean that there is a still better voting system, which nobody has yet invented. It also may be that the "bad 7%" reduces to a much smaller badness-percentage if asset voting is only employed in *multi*winner elections (which we advocate).

Some remarks about Arrow's and others' impossibility theorems. Ken Arrow won the 1972 Nobel prize in Economics for his "impossibility theorem" showing (vaguely speaking) that "good" voting schemes (satisfying a certain list of desiderata) cannot exist. Brams & Fishburn called Arrow's result "one of the most significant developments of the 20th century" in their 2002 survey *Voting procedures* [4].

I believe all that is nonsense. (The time has come to slaughter this sacred cow.) Arrow proved his "impossibility" result by adopting silly rules about what a "voting system" was (his input votes are preference orderings; so is his output). In fact, if the input votes are real vectors and output is identity-of-winner, then one of Arrow's "obviously desirable" postulates actually is obviously *un*desirable and upon fixing that there *is* no Arrow impossibility theorem, as I proved in my paper "Range Voting" [25] by showing that "honest utility voting" satisfies all of Arrow's (revised) postulates. Thus, to put it bluntly, Arrow's "great result" would be better described as "he devised a stupid model."[25]

What I regard as actually a much more important result (although *it* did *not* win a Nobel) is Gibbard's [16] dishonesty theorem saying voters in an N-candidate election, $N \geq 3$, will always sometimes find it strategically desirable to be dishonest. (Actually range voting [25] *partially* avoids that theorem

[23]"Unit mass voting with negotiation" somehow seems to lack zing.

[24]Namely, method 7 of §7.

[25]Also, Arrow required an entire book [1] to prove it (after repairing his original 1950 wrong theorem and proof; the story is recounted in [20]). His theorem and proof were later redone by Fishburn [10] in a few pages and finally by Geanakopolos [15] in 1 page.

in the special case $N = 3$, but not when $N \geq 4$.) Arrow had ignored strategic voting entirely, which was another aspect of his theorem's silliness.

Later, I proved my own impossibility theorems [26][27].

So both of the two original impossibility theorems may be evaded (totally for Arrow, partially for Gibbard) by devising a "voting system" which "breaks the rules," i.e. was not regarded as a "voting system" by the definitions implicit in those theorems.[26] Those original definitions are now seen, with the benefit of hindsight, to have been too restrictive.

The joy (or sorrow) of asset voting is that it "'breaks the rules" in a new way. It is *not* a map from votes to election results at all. Instead it is merely a map from votes to "assets" – which are then used in a "negotiation." Thus in principle the door is now open to devising new voting systems avoiding both Arrow's and Gibbard's "impossibility" theorems and thus potentially better than was previously thought possible. But since "negotiation" is a difficult concept to formalize, it is unclear whether this door really has opened, or whether, upon deeper investigation, it will once again slam shut.

9 Acknowledgements

Brian A. Wichmann spotted a serious misunderstanding of STV by me which had made STV appear much worse than it actually is. In response, the paper was revised to include §6 and to include STV in the experimental study.

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μ	N	D	1:randwin	2:plurS	3:plurH	4:appS	5:rangeH	6:NNass	7:PLass	8:assetH	9:Hstv	10:Sstv
∞	3	1	1.64145	0.14154	0.00000	0.07348	0.00000	0.40894	0.00000	0.00000	0.00000	0.54638
∞	3	2	1.43697	0.55113	0.21690	0.25028	0.09215	0.14159	0.20352	0.39805	0.13167	0.55113
∞	3	3	1.22732	0.49850	0.22421	0.22886	0.10810	0.11090	0.24419	0.39425	0.15543	0.49850
∞	4	1	1.91436	0.21376	0.00000	0.11889	0.00000	0.76455	0.00000	0.00000	0.00000	0.82178
∞	4	2	1.71422	0.82580	0.32071	0.35887	0.12563	0.16047	0.29455	0.64048	0.18060	0.82580
∞	4	3	1.47310	0.74456	0.32239	0.32036	0.12717	0.10948	0.35506	0.60527	0.20729	0.74456
1	3	1	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
1	3	2	0.18977	0.06053	0.04131	0.04458	0.01858	0.02037	0.02692	0.02982	0.02699	0.06053
1	3	3	0.21058	0.07597	0.03739	0.03194	0.01465	0.01523	0.02916	0.03385	0.02405	0.07597
1	4	1	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
1	4	2	0.21420	0.08473	0.06496	0.05876	0.02142	0.01812	0.04122	0.03704	0.04075	0.08473
1	4	3	0.24682	0.11219	0.06044	0.04398	0.01662	0.01421	0.04416	0.05002	0.03574	0.11219
2	3	1	0.82102	0.07126	0.00000	0.03683	0.00000	0.20526	0.00000	0.00000	0.00000	0.27339
2	3	2	1.05927	0.35557	0.20360	0.27215	0.05095	0.19386	0.09246	0.09629	0.09246	0.35557
2	3	3	1.03258	0.35156	0.21084	0.24632	0.04917	0.17632	0.10198	0.10224	0.09870	0.35156
2	4	1	0.95787	0.10606	0.00000	0.05906	0.00000	0.38321	0.00000	0.00000	0.00000	0.41120
2	4	2	1.22655	0.52279	0.32021	0.39685	0.07020	0.22233	0.14374	0.20247	0.14353	0.52279
2	4	3	1.20120	0.52149	0.33035	0.36850	0.06909	0.18191	0.15818	0.22664	0.15339	0.52149
3	3	1	1.23028	0.10698	0.00000	0.05580	0.00000	0.30761	0.00000	0.00000	0.00000	0.41134
3	3	2	1.37789	0.45887	0.22153	0.29766	0.07287	0.18237	0.11994	0.15402	0.11537	0.45887
3	3	3	1.27542	0.43667	0.20910	0.25978	0.07431	0.14481	0.12996	0.16297	0.12029	0.43667
3	4	1	1.43638	0.15954	0.00000	0.08887	0.00000	0.57370	0.00000	0.00000	0.00000	0.61463
3	4	2	1.58701	0.66680	0.35690	0.43024	0.10038	0.21417	0.18218	0.28368	0.17352	0.66680
3	4	3	1.47592	0.63785	0.33477	0.38013	0.09957	0.15694	0.19251	0.29784	0.17587	0.63785
4	3	1	1.43575	0.12484	0.00000	0.06445	0.00000	0.35905	0.00000	0.00000	0.00000	0.47911
4	3	2	1.49265	0.50303	0.22556	0.29250	0.08137	0.17207	0.13584	0.19979	0.12403	0.50303
4	3	3	1.35133	0.46946	0.21103	0.25363	0.08233	0.13463	0.14428	0.20162	0.12764	0.46946
4	4	1	1.67400	0.18584	0.00000	0.10492	0.00000	0.66958	0.00000	0.00000	0.00000	0.71635
4	4	2	1.72303	0.73162	0.35835	0.42067	0.11097	0.19902	0.20187	0.35070	0.18198	0.73162
4	4	3	1.56888	0.68589	0.33144	0.36886	0.10828	0.14552	0.20977	0.35327	0.18227	0.68589
5	3	1	1.53894	0.13401	0.00000	0.06961	0.00000	0.38487	0.00000	0.00000	0.00000	0.51257
5	3	2	1.52671	0.52327	0.22466	0.28318	0.08442	0.16473	0.14725	0.23558	0.12708	0.52327
5	3	3	1.37424	0.48624	0.21333	0.24782	0.08672	0.12899	0.15674	0.23007	0.13231	0.48624
5	4	1	1.79243	0.20038	0.00000	0.11185	0.00000	0.71812	0.00000	0.00000	0.00000	0.77022
5	4	2	1.77213	0.76456	0.35380	0.40806	0.11507	0.18956	0.21554	0.40618	0.18392	0.76456
5	4	3	1.60175	0.71113	0.33004	0.35845	0.11186	0.13885	0.22563	0.39536	0.18627	0.71113

Figure 9.1. Bayesian regrets for 10 single-winner voting systems with 7 voters. (Smaller regrets are better.) Each datapoint is an average over 2,500,000 simulated elections. See the text of §7 for descriptions of the 10 voting systems and of the (μ-clump D-dimensional) probability distributions governing the 7 voters, the N candidates, and their utilities from each other's point of view. The 99% confidence error bars on each regret value should be ± 0.005 or smaller. ▲

μ	N	D	1:randwin	2:plurS	3:plurH	4:appS	5:rangeH	6:NNass	7:PLass	8:assetH	9:Hstv	10:Sstv
∞	3	1	6.01656	0.51442	0.00000	0.26137	0.00000	6.02019	0.00000	0.00000	0.00000	2.02020
∞	3	2	5.42690	2.16125	2.64467	0.91954	0.26901	2.19472	2.59917	3.63132	0.78950	2.16125
∞	3	3	4.62092	1.95052	2.11992	0.84315	0.31616	1.19052	2.84726	3.12186	0.86210	1.95052
∞	4	1	7.04712	0.79609	0.00000	0.44346	0.00000	4.02696	0.00000	0.00000	0.00000	3.03111
∞	4	2	6.44689	3.17011	3.14124	1.25427	0.20153	3.05011	3.46265	4.01366	1.13853	3.17011
∞	4	3	5.55364	2.88727	2.75961	1.12965	0.24771	1.55072	3.39147	2.99895	1.21033	2.88727
1	3	1	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
1	3	2	2.31490	0.68275	0.68187	0.41731	0.23558	0.25742	0.24374	0.42517	0.20755	0.68275
1	3	3	2.44679	0.83019	0.39035	0.15936	0.07222	0.09936	0.24722	0.51125	0.11568	0.83019
1	4	1	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
1	4	2	2.65183	1.01244	1.15456	0.57437	0.22475	0.22613	0.51631	0.35763	0.40907	1.01244
1	4	3	2.92865	1.31540	0.75085	0.25013	0.08706	0.11125	0.41215	0.47674	0.19158	1.31540
2	3	1	3.02114	0.25894	0.00000	0.13678	0.00000	2.99847	0.00000	0.00000	0.00000	1.00553
2	3	2	8.20734	3.24094	4.86640	2.65586	1.04553	3.69536	2.24003	1.79457	2.22254	3.24094
2	3	3	8.21943	3.46955	5.24408	2.44548	1.00252	3.62689	2.59016	1.93289	2.53217	3.46955
2	4	1	3.50993	0.38860	0.00000	0.21863	0.00000	2.01594	0.00000	0.00000	0.00000	1.50006
2	4	2	9.59556	4.61790	6.50954	3.59752	0.90580	3.24904	3.54203	3.18043	3.47029	4.61790
2	4	3	9.73793	4.97283	6.79950	3.33448	0.87367	3.34690	3.98917	3.39668	3.85986	4.97283
3	3	1	4.50858	0.39700	0.00000	0.20679	0.00000	4.51315	0.00000	0.00000	0.00000	1.51157
3	3	2	12.90874	4.35668	4.73412	2.69079	0.78248	3.02204	2.03887	1.86889	1.90001	4.35668
3	3	3	12.40685	4.24489	3.93247	2.19649	0.59021	1.76104	1.92839	1.78924	1.73327	4.24489
3	4	1	5.26983	0.57886	0.00000	0.31660	0.00000	3.02043	0.00000	0.00000	0.00000	2.24368
3	4	2	14.92060	6.38689	8.34035	3.85357	0.69741	4.11684	3.50124	2.95619	3.22026	6.38689
3	4	3	14.44695	6.29120	7.79925	3.19526	0.51438	2.92660	3.26809	2.76481	2.89888	6.29120
4	3	1	5.29342	0.45242	0.00000	0.23199	0.00000	5.27097	0.00000	0.00000	0.00000	1.75233
4	3	2	13.72700	4.58349	4.31436	2.43298	0.74390	2.55893	2.06640	2.45601	1.75599	4.58349
4	3	3	12.84877	4.39684	3.55071	1.98823	0.64832	1.69466	2.03314	2.63892	1.62872	4.39684
4	4	1	6.14536	0.68879	0.00000	0.38067	0.00000	3.50410	0.00000	0.00000	0.00000	2.63580
4	4	2	15.90849	6.74479	7.60583	3.45000	0.63553	3.64580	3.49158	3.47161	2.90649	6.74479
4	4	3	14.98539	6.48117	6.21270	2.83337	0.55491	2.25441	3.29264	3.39748	2.57819	6.48117
5	3	1	5.65726	0.48839	0.00000	0.25533	0.00000	5.63600	0.00000	0.00000	0.00000	1.87368
5	3	2	13.39278	4.48213	4.08632	2.15377	0.66859	2.39055	2.12131	2.97955	1.64104	4.48213
5	3	3	12.70132	4.32411	3.50702	1.78910	0.60266	1.58423	2.10140	3.09617	1.57751	4.32411
5	4	1	6.56193	0.72325	0.00000	0.40172	0.00000	3.75564	0.00000	0.00000	0.00000	2.80731
5	4	2	15.52413	6.55825	6.89255	3.04145	0.56773	3.38481	3.52514	4.00830	2.69461	6.55825
5	4	3	14.78633	6.39985	5.91592	2.55120	0.50580	2.11165	3.34401	3.75977	2.50231	6.39985

Figure 9.2. Bayesian regrets for 10 voting systems with 101 voters. Each datapoint averages 500,000 simulated elections. ▲