Fourth Workshop on
Physics and Computation
Boston University, 22–24 Nov. 1996.

In: T. Toffoli, M. Biafore, and J. Leão (ed.), *PhysComp96*, New England Complex Systems Institute (1996), 208–211. Also on-line in the *InterJournal*.

The maximum speed of dynamical evolution

Norman Margolus[*]
Boston University Center for Computational Science
and MIT Laboratory for Computer Science
Lev B. Levitin
Boston University Department of Electrical and Computer Engineering

Quantum mechanics tells us how to count the number of distinct states that an isolated physical system can be in, and thus allows us to quantify the *maximum amount of information* that can be encoded in this physical system. We discuss here the related problem of quantifying the maximum number of distinct states that such a system can pass through in a given period of time—the *maximum speed of dynamical evolution*. We will conclude that this rate is governed by the average energy E of the system.

Previous analyses have given bounds in terms of the standard deviation ΔE, but ΔE can be arbitrarily large for a fixed value of E. We show here that, given a fixed Hamiltonian and a fixed average energy E, no state can evolve into an orthogonal state in a time less than $h/4(E - E_0)$, where E_0 is the ground state energy of the system. We also discuss the maximum rate for long evolutions, and bounds on information processing rates implied by bounds on the speed of dynamical evolution. For example, adding one Joule of energy to a given computer can never increase it's processing rate by more than about 3×10^{33} operations per second.

1 Introduction

In the realm of computation, the first two quantitative questions that one is likely to ask about a machine are *(i) How much memory does it have?* and *(ii) How fast does it run?* In exploring the computational limits of physical dynamics, one might try to ask the same questions about an arbitrary physical system.[1]

Question *(i)* essentially asks, "How many distinct states can my system be put into, subject to whatever physical constraints I have?" This is really a very old question: the correct counting of physical states is the problem that led to the introduction of Planck's constant into physics[8], and is the basis of all of quantum statistical mechanics. This question can be answered by a detailed quantum mechanical counting of distinct (orthogonal) states. It can also be well approximated in the macroscopic limit[12] by simply computing the volume of phase space available to the system, in units where Planck's constant is 1.

Question *(ii)* will be the focus of this paper. This question can be asked with various levels of sophistication. Here we will discuss a particularly simple measure of speed: the maximum number of distinct states that the system can pass through, per unit of time. For a computer, this would correspond to the maximum number of operations per second.

For a quantum system, the notion of distinct states is well defined: two states are distinct if they are orthogonal. The connection between orthogonality and rate of information processing has previously been discussed[9, 2, 6, 10, 11, 4], but no universal bound was proposed. The minimum time needed for a quantum system to pass from one orthogonal state to another has also previously been characterized, in terms of ΔE[7, 1, 15]. This bound places no limit, however, on how fast a system with bounded average energy can evolve (since ΔE can be arbitrarily large). Here we will, instead, give a universal bound directly in terms of E, the average energy of the system.[2] For a macroscopic system with an unbounded spectrum, this bound is achievable: adding energy increases the maximum rate at which such a system can pass through orthogonal states by a proportionate amount.

2 Maximum rate of dynamics

In the energy basis, quantum time evolutions are constructed out of superpositions of frequency components. One might expect from this that, given a maximum energy eigenvalue, the frequency with which states can change should be bounded by

$$\nu_\perp \leq \frac{E_{\max}}{h} \tag{1}$$

If we take our zero of energy at the ground state of the system,[3] and consider long evolutions, then this relation is true, as we will discuss below. We will also show that, given a fixed average energy E (rather than a fixed maximum), there is a similar bound

$$\nu_\perp \leq \frac{2E}{h}, \tag{2}$$

where again we have taken our zero of energy at the ground state. This equation has the following interpretation: in appropriate units, the average energy of a macroscopic system is numerically equal to the maximum number of orthogonal

[*] Supported by NSF grant DMS-9596217 and by DARPA contract DABT63-95-C-0130

[1] Although many of the computing and communications properies of quantum systems are novel[14, 16, 3], basic questions about memory and speed seem to be rather independent of such considerations.

[2] An information processing bound based on E was proposed by Bremermann[5]; counterexamples to it are given in [9].

[3] In all of this discussion, if for some reason no energies below some $E_{\min}$ are allowed, we should take our zero of energy there instead.

states that the system can pass through per unit of time. This is the maximum rate that can be sustained for a long evolution—the rate at which a system can oscillate between two states is twice as great.

2.1 Orthogonality time

We begin our analysis by discussing the question of the minimum time needed for *any* state of a given physical system to evolve into an orthogonal state.

An arbitrary quantum state can be written as a superposition of energy eigenstates

$$|\psi_0\rangle = \sum_n c_n |E_n\rangle \tag{3}$$

Here and throughout this paper we assume that our system has a discrete spectrum, and that the states have been numbered so that the energy eigenvalues $\{E_n\}$ associated with the states $\{|E_n\rangle\}$ are non-decreasing. To simplify formulas, we will choose our zero of energy so that $E_0 = 0$.

Let $\tau_\perp$ be the time it takes for $|\psi_0\rangle$ to evolve into an orthogonal state. We will now show that, with a fixed average energy E, it is always true that

$$\tau_\perp \geq \frac{h}{4E}. \tag{4}$$

This result is somewhat surprising, since earlier results[7, 1, 15] gave a bound only in terms of ΔE

$$\tau_\perp \geq \frac{h}{4\Delta E} \tag{5}$$

This would suggest that, given a fixed average energy, one could construct a state with a very large ΔE in order to achieve an arbitrarily short $\tau_\perp$. We will now show that this is not the case.

Let us begin by observing that if $|\psi_0\rangle$ is evolved for a time t it becomes

$$|\psi_t\rangle = \sum_n c_n e^{-i\frac{E_n t}{\hbar}} |E_n\rangle \tag{6}$$

Now we let

$$S(t) = \langle\psi_0|\psi_t\rangle = \sum_{n=0}^{\infty} |c_n|^2 e^{-i\frac{E_n t}{\hbar}} \tag{7}$$

We want to find the smallest value of t such that $S(t)$=0. To do this, we note that

$$\begin{aligned} \mathrm{Re}(S) &= \sum_{n=0}^{\infty} |c_n|^2 \cos(\frac{E_n t}{\hbar}) \\ &\geq \sum_{n=0}^{\infty} |c_n|^2 \left(1 - \frac{2}{\pi}(\frac{E_n t}{\hbar} + \sin(\frac{E_n t}{\hbar}))\right) \\ &= 1 - \frac{2E}{\pi\hbar}t + \frac{2}{\pi}\mathrm{Im}(S), \end{aligned} \tag{8}$$

where we have used the identity $\cos(x) \geq 1 - \frac{2}{\pi}(x + \sin(x))$, valid for $x \geq 0$. But for any value of t for which $S(t) = 0$, both $\mathrm{Re}(S) = 0$ and $\mathrm{Im}(S) = 0$, and so (8) becomes

$$0 \geq 1 - \frac{4Et}{h} \tag{9}$$

Thus the earliest that $S(t)$ can possibly equal zero is when $t = h/4E$, which proves (4).

This bound is achievable if the spectrum of energies includes the energy $2E$ (and is very nearly achievable if the spectrum includes a value very close to this, as we would expect, for example, for any normal macroscopic system). In this case, we let

$$|\psi_0\rangle = \frac{|E_0\rangle + |2E\rangle}{\sqrt{2}} \tag{10}$$

which has average energy E. This evolves in a time $t = h/4E$ into

$$|\psi_t\rangle = \frac{|E_0\rangle - |2E\rangle}{\sqrt{2}} \tag{11}$$

which is orthogonal to $|\psi_0\rangle$. If we evolve for the same interval again, we will be back to $|\psi_0\rangle$: the evolution oscillates between these two orthogonal states. For these states, $\Delta E = E$, and so both of the bounds (4) and (5) are achieved.

2.2 Cycles of orthogonal states

In the discussion above, we have seen that a quantum system with average energy E can be made to oscillate between two orthogonal states with a frequency of $4E/h$. Now we address the question of how fast a quantum system can run through a long sequence of orthogonal states. We will show in this section that, for any evolution that passes through a cycle of N mutually orthogonal states at a constant rate,

$$\tau_\perp \geq \frac{N-1}{N}\frac{h}{2E} \tag{12}$$

We will also show that this bound is achievable. Thus for very long evolutions that form a closed cycle, the transition rate between orthogonal states is only half as great as it is for an oscillation between two states. In the next section, we will show that this long-sequence asymptotic rate is achievable in general for any normal macroscopic system.

We begin by noting that the periodic condition that we impose in this section—that the evolution should exactly cycle after N steps—puts a severe restriction on the energy eigenfunctions that can contribute to our initial state. If $|E_j\rangle$ contributes to $|\psi_0\rangle$, and if our cycle length is τ, then we must have that $E_j\tau/\hbar = 2\pi k_j$ for some integer k_j, which means that $E_j = k_j\varepsilon_1$, where $\varepsilon_1 = h/\tau$. Thus our initial state must have the form

$$|\psi_0\rangle = \sum_{n=0}^{\infty} c_n |n\varepsilon_1\rangle \tag{13}$$

For simplicity, we have not included degenerate energy eigenfunctions in our superposition—adding these would not affect our conclusions.

Let $\tau_\perp = \tau/N$ be the interval between consecutive orthogonal states. Noting that $\varepsilon_1\tau_\perp/\hbar = 2\pi/N$, we see that the state obtained from $|\psi_0\rangle$ after m time intervals of length $\tau_\perp$ is

$$|\psi_m\rangle = \sum_{n=0}^{\infty} c_n e^{-\frac{2\pi i n m}{N}} |n\varepsilon_1\rangle \tag{14}$$

and so

$$\langle\psi_{m'}|\psi_m\rangle = \sum_{n=0}^{\infty} |c_n|^2 e^{\frac{2\pi i n}{N}(m'-m)} \tag{15}$$

Now let $|d_n|^2 = \sum_{l=0}^{\infty} |c_{n+lN}|^2$ be the sum of all coefficients that multiply $e^{2\pi i n(m'-m)/N}$. Then

$$\begin{aligned}\langle\psi_{m'}|\psi_m\rangle &= \sum_{n=0}^{N-1} |d_n|^2 e^{\frac{2\pi i n}{N}(m'-m)} \\ &= \delta_{m'm}\end{aligned} \tag{16}$$

since these states are supposed to be orthogonal. Notice that if we let $|d_n|^2 = 1/N$ for all n's, then this last equality is satisfied. Since there are N different possible values of $m'-m$, (16) constitutes N linearly independent equations with N unknown coefficients, and so this solution is unique.

Now we can calculate the relationship between the average energy and the orthogonality time.

$$E = \varepsilon_1 \sum_{n=0}^{\infty} |c_n|^2 n = \varepsilon_1 \sum_{n=0}^{N-1} \sum_{l=0}^{\infty} |c_{n+lN}|^2 (n+lN) \tag{17}$$

but $\sum_{l=0}^{\infty} |c_{n+lN}|^2 (n+lN) \geq \sum_{l=0}^{\infty} |c_{n+lN}|^2 n = n|d_n|^2$, and since $|d_n|^2 = 1/N$,

$$E \geq \varepsilon_1 \sum_{n=0}^{N-1} \frac{n}{N} = \varepsilon_1 \left(\frac{N-1}{2}\right) \tag{18}$$

Since $\varepsilon_1 = h/\tau = h/N\tau_\perp$, this establishes (12).

Note that if all the c_n's for $n \geq N$ are zero, then the inequality above becomes an equality, and so for

$$|\psi_0\rangle = \sum_{n=0}^{N-1} \frac{1}{\sqrt{N}} |n\varepsilon_1\rangle \tag{19}$$

the bound of (12) is achieved. In this case, we combine exactly N energy eigenfunctions to construct a state who's evolution passes through N mutually orthogonal states at the maximum rate.[4]

3 Long sequences of orthogonal states

Now we turn to the question of whether normal macroscopic physical systems can also run through long sequences of orthogonal states with $\tau_\perp = h/2E$. We will show by construction that they can. As in the discussion above, we will not need to use arbitrarily large eigenvalues to achieve this rate, and so our state can be written

$$|\psi_0\rangle = \sum_{n=0}^{N-1} c_n |E_n\rangle \tag{20}$$

[4]If we take the $|n\varepsilon\rangle$'s to be eigenfunctions of the one-dimensional harmonic oscillator, then the orthogonal states that evolve from $|\psi_0\rangle$ of (19) are well known[13].

where E_{N-1} is the maximum energy eigenvalue. Now let

$$c_n = \sqrt{\frac{E_{n+1}-E_n}{E_N}}. \tag{21}$$

This definition of c_n generalizes our result from the previous section: for the special case of $E_n = n\varepsilon_1$, (20) reduces to (19), which achieves $\tau_\perp = h/2E$ in the macroscopic limit. Notice also that, with this definition of c_n, degenerate energy eigenvalues do not appear in our superposition (they get a coefficient of zero, since the E_n's are numbered in non-decreasing order). If there are M distinct eigenvalues used in this superposition, there will be M non-zero c_n's. This definition of c_n always gives normalized states, since

$$\langle\psi_0|\psi_0\rangle = \sum_{n=0}^{N-1} \frac{E_{n+1}-E_n}{E_N} = 1 \tag{22}$$

We can calculate the average energy in the state $|\psi_0\rangle$. This is just

$$\begin{aligned}\langle\psi_0| H |\psi_0\rangle &= \sum_{n,n'} c_{n'}^* c_n E_n \langle E_{n'}|E_n\rangle \\ &= \sum_{n=0}^{N-1} \frac{E_{n+1}-E_n}{E_N} E_n\end{aligned} \tag{23}$$

For $N \gg 1$ and $c_n \ll 1$, we can approximate this sum by an integral. Letting $x = n/N$ and $\varepsilon(x) = E_n/E_N$, we have[5]

$$\begin{aligned}\langle\psi_0| H |\psi_0\rangle &\approx E_N \int_0^1 \varepsilon \frac{d\varepsilon}{dx} dx \\ &= \frac{E_N}{2} \int_0^1 \frac{d}{dx}\varepsilon^2 dx \\ &= \frac{E_N}{2}.\end{aligned} \tag{24}$$

Thus, with this definition of c_n, by giving equal weight to every energy interval we get an average energy that is half of the maximum energy, just as in the limiting case considered in (19).

Now the state obtained from $|\psi_0\rangle$ after m intervals of length $\tau_\perp = h/2E = h/E_N$ is

$$|\psi_m\rangle = \sum_n c_n e^{-2\pi i m \frac{E_n}{E_N}} |E_n\rangle \tag{25}$$

and so

$$\begin{aligned}\langle\psi_{m'}|\psi_m\rangle &= \sum_{n,n'} c_{n'}^* c_n e^{2\pi i \frac{1}{E_N}(m'E_{n'} - mE_n)} \langle E_{n'}|E_n\rangle \\ &= \sum_{n=0}^{N-1} \frac{E_{n+1}-E_n}{E_N} e^{2\pi i \frac{E_n}{E_N}(m'-m)}\end{aligned} \tag{26}$$

As we've seen, this is equal to 1 if $m' = m$. Let $k = m'-m$, and again let $x = n/N$ and $\varepsilon(x) = E_n/E_N$. Then

$$\begin{aligned}\langle\psi_{m'}|\psi_m\rangle &\approx \int_0^1 \frac{d\varepsilon}{dx} e^{2\pi i k\varepsilon} dx \\ &= \int_0^1 \frac{\frac{d}{dx} e^{2\pi i k\varepsilon}}{2\pi i k} dx \\ &= 0.\end{aligned} \tag{27}$$

Thus we can run through a long sequence of orthogonal states at the rate $\nu_\perp = 2E/h = E_{\max}/h$.

[5]All of the other moments can be calculated in a similar manner.

4 Interpretation

For a single isolated system s with average energy $E^{(s)}$, we have seen that we can construct a state that evolves at a rate $\nu_\perp = 2E^{(s)}/h$. If we had many completely independent subsystems, we would have an average energy for the combined system of $E_{\text{tot}} = \sum_s E^{(s)}$. Our construction applies equally well to this composite system, and lets us construct a state for the combined system that evolves at a rate of $\nu_\perp = 2E_{\text{tot}}/h = \sum_s 2E^{(s)}/h$. Thus if we subdivide our total energy between separate subsystems, the maximum number of orthogonal states per unit time for the combined system is just the sum of the maximum number for each subsystem taken separately.

To the extent to which various pieces of a larger system can be approximated as isolated systems for short intervals of time, this analysis applies to interacting systems as well. This is analogous to the case in a parallel computer, where the total number of operations per second for the whole machine is just the sum of the number of operations per second performed by the various pieces. Our result should be interpreted in a similar manner.

Consider for example a system composed of many atoms. It would presumably be more appropriate to calculate a maximum information processing rate by adding up the maximum rates for each atom, than by assuming that all atoms can interact with all other atoms simultaneously at the rate given by the total energy. Since our analysis in this paper hasn't talked about constraints of spatial locality, we can't say much more than this here. We have a bound on the number of operations per second that a computer can perform, but no clear idea of how powerful an operation can be. Presumably, though, what we call an operation in an ordinary computer today is more powerful than the smallest observable change—a change that would make two states orthogonal.

5 Discussion

This non-relativistic analysis says that the average energy (above the ground state) of a macroscopic physical system tells us directly the maximum number of orthogonal states that the system can pass through per unit of time. Just as maximum entropy tells us the potential memory storage of an isolated physical system, average energy tells us its maximum *processing rate*.

Acknowledgments

Margolus would like to thank Ed Fredkin, who first made him aware of the question, "How can we physically quantify computation." He and Tom Toffoli have encouraged Margolus and inspired him in his studies of the mechanics of information in physics. Margolus would also like to thank Mark Smith and Sandu Popescu for useful comments on this work, Tom Knight and Gerry Sussman, for their support and encouragement, and all of these people for useful discussions.

Both authors would like to thank the Institute for Scientific Interchange for the opportunity to work together on this research at their 1996 Quantum Computing Workshop in Turin, Italy.

References

[1] Anandan, J., and Y. Aharonov, "Geometry of quantum evolution," *Phys. Rev. Lett.* **65** (1990), 1697–1700.

[2] Benioff, P. A., "Quantum mechanical Hamiltonian models of discrete processes that erase their own histories: application to Turing machines," *Int. J. Theor. Physics* **21** (1982), 177–202.

[3] Bennett, C. H., "Quantum Information and Computation," *Physics Today* (October 1995), 24–30.

[4] Biafore, M, "Few-body cellular automata," MIT Ph.D. Thesis (1993), published as MIT/LCS/TR-597 (MIT Laboratory for Computer Science, 1993).

[5] Bremermann, H. J., "Quantum noise and information," in *Proceedings of the 5th Berkeley Symposium on Mathematical Statistics and Information,* (University of California Press, Berkeley, Vol. 4, 1967), 15–22.

[6] Feynman, R. P., "Quantum mechanical computers," *Opt. News* **11** (1985).

[7] Fleming, G. N., *Nuovo Cimento A* **16**, (1973), 232–240.

[8] Klein, M. J. "Planck, Entropy, and Quanta, 1901-1906," *The Natural Philosopher* **1** (1963), 83-108.

[9] Levitin, L. B., "Physical limitations of rate, depth, and minimum energy in information processing," *Int. J. Theor. Phys.* **21** (1982), 299–309.

[10] Margolus, N., "Quantum computation," *New Techniques and Ideas in Quantum Measurement Theory* (Daniel Greenberger ed.), New York Academy of Sciences (1986), 487–497.

[11] Margolus, N., "Parallel quantum computation," *Complexity, Entropy, and the Physics of Information* (Wojciech Zurek ed.), Addison-Wesley, 1990.

[12] Maslov, V. and M. Fedoriuk, *Semiclassical approximations in quantum mechanics,* Reidel (1981).

[13] Pegg, D. T. and S. M. Barnett, *Phys. Rev. A* **39** (1989), 1665.

[14] Schumacher, B., "Quantum Coding," *Phys. Rev. A* **51** (1995), 2738.

[15] Peres, A., *Quantum Theory: Concepts and Methods*, (Kluwer Academic Publishers, Hingham, MA, 1995).

[16] Shor, P. W. "Algorithms for quantum computation: discrete log and factoring," Proceedings of the 35th Annual Symposium on the Foundations of Computer Science (IEEE Computer Society Press, Los Alamitos, CA, 1994), 124.

[17] Vaidman, L., "Minimum time for the evolution to an orthogonal quantum state," *Am. J. Phys.* **60:**2 (1992), 182–183.