ANOMALOUS OUTCOMES IN PREFERENTIAL VOTING

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ABSTRACT. We consider the preferential voting system and investigate the frequency of non-monotonic allocations of votes, in which by getting more votes, a winning candidate can become a losing candidate.

1. INTRODUCTION

There has been considerable attention given over the last centuries to the problem of finding a voting system that produces outcomes that are genuinely representative of the will of the electorate. In some of the earliest recorded debate on the issue, in the late 18th Century, the French Mathematicians Borda and Condorcet became involved in a debate over the best system to use for election of members to the French Academy of Sciences. They both introduced voting systems, versions of which remain in use today. In fact, large numbers of voting systems and variants are in common use [5, 9].

It is well known that in electoral systems in which one is seeking a winner (or an overall ordering) from $n \geq 3$ candidates, there are a number of desirable conditions that are mutually incompatible. Specifically, Arrow's impossibility theorem [2] shows that the only voting system satisfying *monotonicity* (informally that getting extra votes cannot cause a candidate to lose an election), *independence of irrelevant alternatives* (where if a non-winning candidate is removed and all of her votes are transferred to the next preference of the voter, the winner should still win) and *decisiveness* (where every election has a winner) is a dictatorship!

In spite of these proven incompatibilities, it is of interest to look for a voting systems that avoids these difficulties as much as possible. In particular, it is of interest to consider the frequency of the failure of the various desired conditions.

In the *preferential voting system* (also known as the *alternative vote system*, the *Hare system of voting* or *instant runoff voting*), a simpler version of the more widely used *single transferable vote* introduced by Hare [8], a voter makes a selection from the slate of candidates by indicating an order of preference from his top candidate (marked with a 1) downwards. To count the election, the first place votes for each candidate are counted up and the candidate with the fewest first choice votes is eliminated. Each vote for that candidate is then transferred to the candidate indicated as the next preference. This process is repeated until a single candidate remains. This candidate is then declared the winner. This system is used in Australian Elections for the House of Representatives as well as in City Council elections in Cambridge, Massachusetts. Recently, the system has been adopted in

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San Francisco. It is being considered for implementation in statewide elections in Alaska and Vermont.

The preferential voting system (and also the single transferable vote or STV system) has a drawback, which is that it fails to be monotonic. We formally define this property as follows. Suppose for some particular assignment of votes, candidate X is the winner. Further suppose the votes are subsequently modified by improving on certain ballots the position of X relative to the other candidates but without changing the relative position of any other pair of candidates on any ballot. A voting system is called *monotonic* if X is **necessarily** still be the winner using the modified ballots.

The following example demonstrates that the preferential voting system fails to be monotonic. In the first election, the initial proportions of the vote are A: 41%, B: 30% and C: 29%. When C is eliminated, the votes marked CAB are redistributed to A, whereas the votes marked CBA are redistributed to B. Once this is done, A has 55% of the vote, whereas B has 45% of the vote so A is the winner.

	ABC	ACB	BAC	BCA	CAB	CBA
Election 1	20	21	5	25	14	15
Election 2	22	21	5	23	14	15

In the second election, the initial proportions of the vote are A:43%, B:28% and C:29%. After B is eliminated and the votes are redistributed, A has 48% of the vote, whereas C has 52% of the vote so that C is the winner.

One notes though that the only difference between the two elections is that 2% of the electorate has switched its vote from BCA to ABC from the first election to the second. Thus by gaining more of the vote, A has become a loser!

While the lack of monotonicity in the voting system is fairly well-known, at least in the technical literature, this information does not appear to have entered the public discussion of this voting system. In a report [4] on voting systems to the British House of Lords, commisioned by the Electoral Reform Society, it was stated (based on articles of Allard [1] and Bradley [3]):

> STV has been criticised for not being monotonic in certain circumstances. This, however, is a theoretical rather than a practical difficulty.

The article continued with two assertions:

(1) It has been shown that in the UK elections non-monotonicity would only occur once every century.

(2) Moreover, non-monotonicity has never been demonstrated in any STV election.

The purpose of this article is to question these two assertions and to draw a wider conclusion, namely that under the models that we will study, not only is non-monotonicity possible, it is overwhelmingly likely if there are large numbers of candidates.

2. Description of the models and Statement of results

In a non-monotonic system, by definition, some allocations of votes can be modified as described above in favor of the winner, with the effect that the winner then loses the election. Such an allocation of votes will be said to be *non-monotonic*.

In this paper, we will discuss the frequency of non-monotonic allocations under two models of voting.

For simplicity, we assume that all voters rank all candidates. To describe the first model, we note that if there are n candidates, there are $n!$ possible orderings of the candidates. The election is completely described by knowing the proportion of voters with each of the $n!$ preference lists. Thus we can view vote allocations as points in an $n! - 1$-dimensional simplex. In the *simplex model*, an assumption is made that vote allocations are uniformly distributed on the simplex. Let $\mathcal{P}^n$ denote the simplex of possible allocations of votes. The first model therefore corresponds to the normalized Lebesgue measure on the simplex, which we shall denote by $\mathbb{P}^n_{\mathrm{SM}}$.

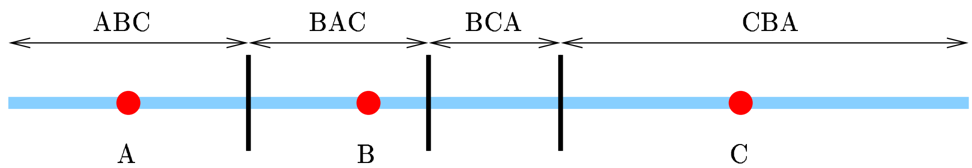

FIGURE 1. Political Spectrum Model

The second model is called the *political spectrum model*. In this model, the political spectrum is the unit interval $[0,1]$ and the n candidates are assumed to be located at independent uniformly distributed points of the political spectrum. Voters correspond to the continuum of points in $[0,1]$ and they are also assumed to be uniformly distributed. A particular voter x lists candidates in order of increasing distance from x.

In this way, the interval is divided into $\binom{n}{2}+1$ subintervals of voters with identical preference lists. The proportion of voters with a given preference list is then given by the measure of the subinterval with that preference list. In this way, our second model specifies a second measure on the simplex.

We denote this measure by $\mathbb{P}^n_{\mathrm{PS}}$. We note that the measure is highly singular since it is supported on a collection points for which at most $\binom{n}{2} + 1$ of the $n!$ coordinates are non-zero.

The model is illustrated in Figure 1. Here the candidates are A, B and C and the voters with each of the preference lists ABC, BAC, BCA and CBA are marked on the figure (preference lists ACB and CAB are impossible as for example any voter with first choice A would prefer B to C). The proportion of voters with a given list is given by the length of the interval corresponding to the voters with that particular list.

If we let N^n denote the set of vote allocations on $\mathcal{P}^n$ which are non-monotonic, then our task is to estimate the probabilities $\mathbb{P}^n_{\mathrm{SM}}(N^n)$ and $\mathbb{P}^n_{\mathrm{PS}}(N^n)$.

Theorem 1. *We have* $\mathbb{P}^n_{SM}(N^n) = 1 - O(n^{-(1-\epsilon)})$.

We remark that we have calculated $\mathbb{P}^n_{\mathrm{SM}}(N^n)$ and $\mathbb{P}^n_{\mathrm{PS}}(N^n)$ in the case $n = 3$, where the values were respectively $13/288$ (4.5%) and $5/72$ (6.9%). These values were the result of explicit integration over the region of non-monotonicity mentioned above.

In particular, it is not hard to show that in the political spectrum model with three candidates, there is a non-monotonic allocation of votes if and only if the central candidate gets at least 25% of the vote and is the first to be eliminated. To see this, note that in this case, the non-winning outer candidate can give the winner enough of her vote to have *slightly* less of the vote than the central candidate. This candidate is then eliminated (now having the smallest share of the vote) and all of her votes go to the central candidate rather than to the more extreme winner, allowing the central candidate to become the new winner.

As an application of this, one can examine British electoral data from 1983. In the South Derbyshire seat, there were three candidates: Conservative (43.8%), Labour (29.2%) and SDP (27.0%). The SDP was certainly the central candidate and crossover from Labour to Conservative seems very unlikely. Given this, we see that had the preferential voting system been in use, this result would almost certainly have been non-monotonic. While it is not possible based on the recorded information (or even based on the actual ballot papers) to conclude with certainty what would have happened in the preferential voting system, it strongly suggests that if one knows what to look for, there would not be a problem in finding a counterexample to Assertion (2) above. Similar examples in the same year's election appear to contradict Assertion (1) also.

We note that in the political spectrum model, we have so far been unable to demonstrate rigorously that $\mathbb{P}^n_{\mathrm{PS}}(N^n) \to 1$, but there is considerable computer evidence that this is the case. The main difficulty is that there are many different ways in which non-monotonicity can occur. One simple criterion by which one can detect that a candidate will not be elected is if her vote share is a 'local minimum' vote share on the political spectrum at some stage (i.e. if she has a smaller proportion of the vote at some stage than either of her neighbours). Accordingly the idea is to perturb the votes such that at some stage, the winning candidate becomes a local minimum. Computer evidence suggests that the probability of being able to do this increases rapidly to 1 as n increases. Since this is only one mode of inducing non-monotonicity and a fairly crude one at that, it seems likely that the true percentage is significantly higher than is currently being detected by the simulations.

3. Condorcet winners

Our strategy for the proof of Theorem 1 will involve estimating the probability that there is a Condorcet winner. A *Condorcet winner* is a candidate who would beat any other candidate in a one-to-one contest, that is a candidate X with the property that for any other candidate Y, more than 50% of the voters put X higher than Y on their preference lists. The well-known Condorcet paradox states that there need not be a Condorcet winner in an election.

We will show that in the simplex model, with probability approaching 1 as the number of candidates tends to infinity, there is no Condorcet winner. Given this, our approach to showing non-monotonicity will be as follows. Since it can be assumed that there is no Condorcet winner, there will be a candidate who would prevail in a one-to-one contest with the winner. The idea is then to modify the votes in favor of the winner to ensure that all the candidates apart from the winner and the second candidate are eliminated so that the actual winner and the second candidate are compared head-to-head. At this point, the winner is defeated.

By contrast, it is not hard to see that in the political spectrum model, there is almost surely a Condorcet winner, namely the candidate closest to the center of the interval. (More generally if the voters are assumed to be distributed according to some other distribution, the Condorcet winner is the candidate closest to the median position on the spectrum of the voters).

Theorem 2. *If we let C^n be the collection of vote allocations in $\mathcal{P}^n$ for which there is a Condorcet winner, then $\mathbb{P}^n_{SM} = O(n^{-(1-\epsilon)})$.*

In the proof, we shall make use of the following simple theorem.

Lemma 3. *Suppose that X and X' are two random variables taking values in a measurable space Z with probability distributions $\mathbb{P}$ and $\mathbb{P}'$. Let λ be a measure on Z such that $\mathbb{P}$ and $\mathbb{P}'$ are both absolutely continuous with respect to λ and write ρ and ρ' for the densities of $\mathbb{P}$ and $\mathbb{P}'$ with respect to λ.*

If for some $0 < \epsilon < 1$ and constant C, there exists a set $A \subset Z$ such that $\mathbb{P}(A) > 1-\epsilon$, $\mathbb{P}'(A) > 1-\epsilon$ and $|\log \rho - \log \rho' - C| < \epsilon$ on A, then the total variation distance between $\mathbb{P}$ and $\mathbb{P}'$ is at most 23ϵ.

Proof. On A, we have $e^{C-\epsilon}\rho'(x) < \rho(x) < e^{C+\epsilon}\rho'(x)$. Integrating this inequality over A, we recover the two inequalities $e^{C-\epsilon}(1-\epsilon) < 1$ and $1-\epsilon < e^{C+\epsilon}$. In particular, $(1-\epsilon)e^{-\epsilon} < e^C < e^{\epsilon}/(1-\epsilon)$. It then follows that $e^{-3\epsilon} < \rho(x)/\rho'(x) < e^{3\epsilon}$ on A so that $|\rho(x) - \rho'(x)| < (e^{3\epsilon}-1)\rho'(x)$ on A. Now integrating $|\rho(x) - \rho'(x)|$ over the space, we get a bound of 2ϵ on A^c and $e^{3\epsilon} - 1 < 21\epsilon$ on A. The result follows. $\square$

Proof of Theorem 2. There are two steps to the proof. First, after restricting the orders given by the voters to orderings on the first $\lfloor n/9 \rfloor$ candidates, it is shown that the resulting distribution is close in total variation distance to a multivariate normal distribution. Second, the probability of non-monotonicity is estimated in the easier case of the multivariate normal distribution.

If the candidates are numbered from 1 to n, we shall take $m = \lfloor n/9 \rfloor$ and estimate the probability that 1 is a Condorcet winner when the votes are restricted to orderings of the first m candidates. Since if 1 is a Condorcet winner overall, she is automatically a Condorcet winner in the restricted election, this will allow us to get an upper bound on the probability of there being a Condorcet winner overall.

Clearly when there are m candidates under consideration, there are $m!$ possible orders. Let the orders be $\pi_1, \ldots, \pi_{m!}$. The restricted vote allocation is a point on the $m!-1$-dimensional simplex $\sum_{i=1}^{m!} X_i = 1$, where X_i represents the proportion of the electorate with restricted order π_i. We shall need to study the induced distribution on that space.

By assumption, the probability density function on the full $n!-1$-dimensional simplex is given by $\rho_0(u_1, \ldots, u_{n!-1}) = C$, in the region $\sum U_i \le 1$. Define $U_{n!} = 1 - (U_1 + \ldots U_{n!-1})$. The U_i represent the proportion of the electorate with each of the $n!$ full orders. Since for each ordering of the first m candidates, there are $n!/m!$ possible extensions to the entire field of n candidates, we will assume that the U_i are numbered so that the jth block of $n!/m!$ U variables corresponds to those orderings in which amongst the first m candidates, the ordering is π_j. This gives $X_j = U_{(j-1)n!/m!+1} + \ldots + U_{j\,n!/m!}$ for $j \le m!$. Given this, we can change variables on the $n!-1$-dimensional simplex to (X_j) (where j runs from 1 to $m!-1$) along with $(U_{(j-1)n!/m!+i})$ (where j runs from 1 to $m!$ and i runs from 1 to $n!/m!-1$). With respect to these variables, the simplex is defined by the inequalities $U_{(j-1)n!/m!+1} + \ldots + U_{j\,n!/m!-1} \le X_j$ (for $1 \le j \le m!$) and $X_1 + \ldots + X_{m!-1} \le 1$. Since the Jacobian of the change of variables is 1, the numerical value of the density remains the same under this transformation. Integrating over the u_j, we recover the joint probability density function of the marginal distribution of $(X_1, \ldots, X_{m!-1})$:

$$\rho(x_1, \ldots, x_{m!-1}) = Cx_1^{n!/m!-1} \ldots x_{m!-1}^{n!/m!-1}(1 - x_1 - \ldots - x_{m!-1})^{n!/m!-1},$$

where C is a normalizing constant. If we write $x_{m!}$ for $1 - (x_1 + \ldots + x_{m!-1})$, then this may be more simply expressed as

$$\rho(x_1, \ldots, x_{m!-1}) = Cx_1^{n!/m!-1} \ldots x_{m!}^{n!/m!-1}.$$

It is helpful to observe that each of the variables X_i $(1 \leq i \leq m!)$ have beta distributions with parameters $n!/m!$ and $n! - n!/m!$. In other words, they have a density on $[0, 1]$ given by $Cx^{n!/m!-1}(1 - x)^{n!-n!/m!-1}$ for a normalizing constant C. One can then calculate the mean and variance of these variables to be $1/m!$ (as expected) and $(m! + 1)/(m!^2(n! + 1))$ (approximately $1/(m!n!)$).

We now construct affine copies Y_i of the X_i, setting the mean to 0 and the variance to approximately 1. Specifically, we define $Y_i = \sqrt{n!m!}(X_i - 1/m!)$ (for $1 \leq i \leq m!$). Note that $\sum_i Y_i = 0$. As before, we write $y_{m!}$ for the expression $-(y_1 + \ldots y_{m!-1})$. When expressed with respect to the variables (Y_i), the probability density becomes

$$\begin{aligned}
&\log \rho(y_1, \ldots, y_{m!-1}) \\
=&C + \left(\frac{n!}{m!} - 1\right)\left(\log\left(\frac{1}{m!} + \frac{y_1}{\sqrt{n!m!}}\right) + \ldots + \log\left(\frac{1}{m!} + \frac{y_{m!}}{\sqrt{m!n!}}\right)\right) \\
=&C + \left(\frac{n!}{m!} - 1\right)\left(\log\left(1 + \sqrt{\frac{m!}{n!}}y_1\right) + \ldots + \log\left(1 + \sqrt{\frac{m!}{n!}}y_{m!}\right)\right),
\end{aligned}$$

where by the C's, we mean in each case different constants. Using Chebyshev's inequality, we estimate that the probability that a given variable Y_i satisfies $|Y_i| > m!$ is $O((m!)^{-2})$. Since there are $m!$ variables (including the remainder term $Y_{m!}$), we calculate that the probability that any one of them has a deviation of this order is $O((m!)^{-1}) = o(e^{-n})$.

We let A be the set where $|Y_i| < m!$ for each i. By the above, this is a set with high probability. Taking a Taylor expansion of the expression for the density above, we see that the first order terms cancel so that we get

$$\log \rho(y_1, \ldots, y_{m!-1}) = C - \frac{1}{2}(y_1^2 + \ldots + y_{m!}^2) + O(m!^{9/2}/n!^{1/2}).$$

It can be seen that this error term is $o(e^{-n})$.

We shall compare the distribution of $(Y_1, \ldots, Y_{m!-1})$ with a multivariate normal distribution $(Z_1, \ldots, Z_{m!-1})$ defined by

$$\log \rho'(z_1, \ldots, z_{m!-1}) = C' - \frac{1}{2}(z_1^2 + \ldots + z_{m!-1}^2 + (z_1 + \ldots + z_{m!-1})^2).$$

In order to apply Lemma 3, it is just necessary to ensure that the sequence $(Z_1, \ldots, Z_{m!-1})$ takes values in the set A with high probability. For details of multivariate normal distributions, the reader is referred to [7]

To establish this, we recover from the above expression the covariance matrix of the variables $(Z_1, \ldots, Z_{m!-1})$ by inverting the matrix in the above bilinear form. Specifically, the inverse of the covariance matrix is given by

$$A^{-1} = \begin{pmatrix} 2 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 1 & \cdots & 1 \\ 1 & 1 & \ddots & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 2 \end{pmatrix}.$$

We therefore see that the covariance matrix of the variables (Z_s) is given by

$$\frac{1}{m!}\begin{pmatrix} m!-1 & -1 & -1 & \cdots & -1 \\ -1 & m!-1 & -1 & \cdots & -1 \\ -1 & -1 & \ddots & \cdots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & m!-1 \end{pmatrix}.$$

Setting $Z_{m!} = -(Z_1 + \ldots + Z_{m!-1})$, we see that the variables Z_i each have variance of $(m!-1)/m!$ (and mean 0) and hence the previous argument shows that with very high probability $(1 - o(e^{-n}))$, $(Z_i) \in A$.

It follows that the total variation distance between the random variables (Y_i) and (Z_i) is $o(e^{-n})$ (that is to say that if we denote by $\mathbb{P}_{\mathrm{MVN}}$ the distribution on $\mathbb{R}^{m!}$ of the random variables (Z_i) and by $\mathbb{P}_Y$ the distribution of (Y_i), then for any event E, $|\mathbb{P}_Y(E) - \mathbb{P}_{\mathrm{MVN}}(E)| = O(e^{-n})$).

Recall that we need to estimate the probability that candidate 1 is a Condorcet winner. Clearly, for candidate 1 to be a Condorcet winner it is necessary that for each j with $1 < j \le m$, $\sum_{\{s:\, 1 >_s j\}} X_s > \frac{1}{2}$, where by the notation $1 >_s j$, it is meant that candidate 1 is preferred to candidate j in the order π_s. Expressed in terms of the variables Y_s, this says simply $\sum_{\{s:\, 1>_s j\}} Y_s > 0$. Letting A be the region in $\mathbb{R}^{m!}$ where these inequalities hold, we are trying to estimate $\mathbb{P}_Y(A)$. As explained above, it is then sufficient to estimate $\mathbb{P}_{\mathrm{MVN}}(A)$ or in other words the probability that $\sum_{\{s:\, 1>_s j\}} Z_s > 0$ for each j with $1 < j \le m$.

From the matrix, we see that for $i < j < m!$, the covariance of Z_i and Z_j is $-1/m!$. It is straightforward to check that this holds also if $j = m!$.

It will be convenient to introduce new variables, W_j, given by

$$W_j = \sqrt{\frac{6}{m!}} \sum_{\{s:\, 1>_s j\}} Z_s,$$

so that it is now sufficient to estimate instead the probability that W_j is positive for all $2 \le j \le m$.

We calculate the variance and covariances of the variables W_i. We see that

$$\mathrm{Var}(W_i) = \frac{6}{m!}\left((m!/2)\frac{m!-1}{m!} + ((m!/2)^2 - m!/2)\frac{-1}{m!}\right) = \frac{3}{2}; \quad \text{and}$$

$$\mathrm{Cov}(W_i, W_j) = \frac{6}{m!}\left((m!/3)\frac{m!-1}{m!} + ((m!/2)^2 - m!/3)\frac{-1}{m!}\right) = \frac{1}{2} \quad \text{for } i \neq j$$

Clearly the variables $(W_j)_{2\le j\le m}$ have a mean zero multivariate normal distribution (as they are a linear combination of mean zero random variables with a multivariate normal distribution). This distribution is therefore defined by the above variances and covariances. In order to estimate the probability that the

W_i are all positive, we introduce a further set of random variables with the same distribution.

Namely, we let N and $N_2, \ldots, N_m$ be independent normal random variables with mean 0 and variance 1. Now forming $V_j = N/\sqrt{2} - N_j$, we see immediately that the variables (V_j) have the same distribution as the variables (W_j) above.

To estimate the probability that the (W_j) are all positive, it is now sufficient to estimate the probability of the event that $N > \sqrt{2}N_j$ simultaneously for each j. Denote this event by S. We observe that for any number α, $S \subset \{N > \alpha\sqrt{2}\} \cup \{N_j < \alpha\colon 2 \le j \le m\}$. This gives the bound

$$\begin{aligned}\mathbf{P}(S) &\le \frac{1}{\sqrt{2\pi}}\int_{\alpha\sqrt{2}}^{\infty} e^{-t^2/2}\,dt + \left(1 - \frac{1}{\sqrt{2\pi}}\int_{\alpha}^{\infty} e^{-t^2/2}\,dt\right)^{m-1} \\ &\le \frac{e^{-\alpha^2}}{2\sqrt{\pi}\alpha} + \left(1 - \frac{e^{-\alpha^2/2}}{\sqrt{2\pi}}\left(\frac{1}{\alpha} - \frac{1}{\alpha^3}\right)\right)^{m-1},\end{aligned}$$

where the last inequality comes from the following bound (see [6], page 166):

$$\frac{e^{-x^2/2}}{\sqrt{2\pi}}\left(\frac{1}{x} - \frac{1}{x^3}\right) \le \frac{1}{\sqrt{2\pi}}\int_x^{\infty} e^{-t^2/2}\,dt \le \frac{e^{-x^2/2}}{\sqrt{2\pi}}\frac{1}{x},$$

valid for $x > 0$.

Setting $\alpha = \sqrt{(2-\epsilon)\log m}$, we see

$$\mathbb{P}(S) \le \frac{1}{2m^{2-\epsilon}\sqrt{(2-\epsilon)\pi\log m}} + \left(1 - \frac{1}{m^{1-\epsilon/3}}\right)^{m-1} = O(m^{-(2-\epsilon)}) = O(n^{-(2-\epsilon)}).$$

Since there are n candidates, the probability that one of them is a Condorcet winner is bounded above by $O(n^{-(1-\epsilon)})$. This completes the proof of the theorem. □

4. Proof of Theorem 1

Proof of Theorem 1. If π_s is an ordering of $1, \ldots, n$, write U_s for the proportion of the people with π_s as their choice. It can be seen as above that the random variables U_s have density $\rho(u) = (n! - 1)(1-u)^{n!-2}$. Similarly, the joint distribution of two of the random variables U_s and U_t is $\rho_2(u, u') = (n!-1)(n!-2)(1-u-u')^{n!-3}$. Using this, we see that variance v of one of the variables is $(n!-1)/(n!^3 + n!^2)$ and the covariance c of two of them is $-1/(n!^3 + n!^2)$. Accordingly, if a new variable Y is formed by taking the sum of any fixed subset (of size r) of the U_s, its variance is given by $rv + r(r-1)c = r(n!-r)/(n!^3 + n!^2) \le 1/(4(n!))$.

For $1 \le i, j, k \le n$ (with $j \ne k$) and a subset B of $\{1, 2, \ldots, n\}$ not containing any of i, j or k, let $V_{i,j,k,B}$ be the sum of the U_s over those s with the property that $k >_s j$ and i is the maximum of π_s restricted to $\{1, \ldots, n\} \setminus B$. Since these variables V are of the form described above, we can estimate that the probability that such a variable differs from its expectation by more than $1/(4n)$ is bounded above by $4n^2/n!$ (using Chebyshev's theorem). Since there are less than $n^3 2^n$ such variables, the probability that any one of them differs from its average value by more than $1/(4n)$ is bounded above by $4n^5 2^n/n! = o(n^{-1})$.

Let S_1 be the event that none of the random variables differs from its expectation by more that $1/(4n)$ and S_2 be the event that there is no Condorcet winner. From Theorem 2, $\mathbb{P}(S_1 \cap S_2) = 1 - O(n^{-1+\epsilon})$. We will show that vote allocations in $S_1 \cap S_2$ are non-monotonic.

To see this, fix an allocation in $S_1 \cap S_2$. We will denote the winner of the election by 1. Since there is no Condorcet winner, there is another candidate preferred to 1 by the majority of the electorate. We will call this candidate 2. We modify the vote in favor of 1 by reassigning all votes that prefer 1 to 2 to candidate 1. Since the majority of the electorate preferred 2, the proportion of votes for candidate 1 is now (slightly) less than one half. All the remaining votes for other candidates have candidate 2 placed higher than candidate 1.

Initially, the votes for the candidates are $\frac{1}{2} - \delta^+$ for candidate 1, $\frac{1}{n} + \delta$ for candidate 2 and $\frac{1}{2n} + \delta$ for the other candidates, where each time δ is used, it denotes an unknown quantity less than $1/(4n)$ in modulus. Similarly, δ^+ is an unknown positive quantity. To justify these assertions, note that the number of votes after adjustment for a candidate i with $2 \leq i \leq n$ is $V_{i,1,2,\emptyset}$ and the quantities $1/n$ and $1/2n$ are simply the expectations of the relevant random variables.

Accordingly, one of the 'other' candidates is removed and his/her votes are redistributed to the subsequent choices of the voters. Note that candidate 1 does not receive any further votes in this redistribution, but that the votes are equally likely to go to any of the remaining candidates.

Accordingly, after the first round of removals is complete, the proportions for the candidates are $\frac{1}{2} - \epsilon$ for candidate 1, $\frac{1}{n} + \frac{1}{2n(n-2)} + \delta$ for candidate 2, and $\frac{1}{2n} + \frac{1}{2n(n-2)} + \delta$ for the others, (the number of votes for a remaining candidate i with $2 \leq i \leq n$ being $V_{i,1,2,\{x\}}$ where x was the candidate eliminated in the first round.

Given that the vote allocation was in S_2, the quantities δ are all bounded above by $1/(4n)$ in modulus so that candidate 2 remains ahead of all the remaining candidates of 3 to n. This means that again, one of the 'others' will be removed. This process continues, with candidate 2 remaining ahead of all the other candidates until only candidates 1 and 2 remain. By assumption, a majority of the electorate prefer candidate 2 to candidate 1, so that candidate 1 (the previous winner) has become a loser by gaining extra votes. This completes the proof.

□

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