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AN EXTENSION OF THE GENERALIZED WEBER PROBLEM†

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1. INTRODUCTION

Several papers have independently presented methods for solving the generalization of the Weber problem, i.e., finding a single point (origin or source) in two-dimensional Euclidean space which is the minimum transport distance and/or cost point for any number of destination points. See Cooper [1], [2], Kuhn and Kuenne [5], Miele [6], and Palermo [7]. Some further extensions of this Weber problem have also been presented by the present author in Cooper [1], [2], and [3].

A mathematical statement of the generalization of the simple Weber problem is as follows.

Let the locations of the set of n known destination or shipping points be given by $(x_j, y_j, j = 1, \ldots, n)$, their co-ordinates in two-dimensional Euclidean space. Let the co-ordinates of the unknown origin or supply point that is to be determined be (x, y). Let $\beta_j, j = 1, \ldots, n$ be positive weights relating to amounts to be shipped (or any other desired weights). Then the problem of finding the point (x, y) such that the sum of distances or costs proportional to distances shall be a minimum can be stated as:

$$(1) \qquad Min\ \phi_1 = \sum_{j=1}^{n} \beta_j \left[(x_j - x)^2 + ((y_j - y)^2\right]^{1/2}$$

In Cooper [1], [2], [3], and Kuhn and Kuenne [5], necessary and sufficient conditions are stated for (x, y) to be a solution to (1). In addition, an iterative numerical technique is described for obtaining the solution. This computational technique is rapid and effective in practice and has been used for the solution of realistically large problems in practice. See Cooper [1], [2], [3].

In this paper we shall be concerned with a modification of the problem stated in (1). The modification concerns the fact that in realistic situations the cost of shipment or servicing may not be simply proportional to distance (with weights). In some situations it would be more appropriate to assume that the costs were proportional to distance raised to some power. This gives greater flexibility in realistically and accurately fitting cost data.

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This modification can be expressed in terms of the following problem.

Find the point in Euclidean space E^2, (x, y) such that the weighted sum of distances to the K^{th} power is a minimum, i.e.,

$$(2) \qquad Min\ \phi = \sum_{j=1}^{n} \beta_j \left[(x_j - x)^2 + (y_j - y)^2 \right]^{K/2}, \quad K > 0$$

In the remaining sections of this paper, a mathematical characterization of this problem will be given. In addition, the results of extensive calculations relating to the solution of (2) will be given.

2. MATHEMATICAL CHARACTERIZATION OF THE EXTENDED PROBLEM

By the "extended problem" we shall mean the solution of (2), i.e., looking for a point (x, y) in E^2 that will yield the global (absolute) minimum of the right-hand side of (2). Before discussing possible solution methods we shall characterize the problem in the following theorems. Proofs are in the Appendix.

Theorem 1: The function $\phi = \sum_{j=1}^{n} \beta_j \left[(x_j - x)^2 + (y_j - y)^2 \right]^{K/2}, \quad K > 0$

is a convex function for $K \geq 1$. If $K < 1$, ϕ is neither convex nor concave.

The proof of Theorem 1 indicates that for values of $K < 1$, the function ϕ may not be convex. Therefore, unlike the case of Equation (1) or Equation (2) when $K > 1$, the presence of local minima, when $K < 1$, is a distinct possibility. The following theorem establishes the fact that at each of the destination points, (x_j, y_j), ϕ has a local minimum.

Theorem 2: The points (x_j, y_j) $j = 1, \ldots, n$ are local minima in the problem:

$$Min\ \phi = \sum_{j=1}^{n} \beta_j \left[(x_j - x)^2 + (y_j - y)^2 \right]^{K/2}, \quad 0 < K < 1$$

We turn now to the calculation of the minimum point (x, y) for the problem of Equation (2). For $K \geq 1$, we know that ϕ is a convex function and therefore every local minimum is a global minimum. If we calculate the first partial derivative, given by Equations (15) and (16) in the Appendix, the condition that these derivatives be zero is both necessary and sufficient for the existence of a minimum. We therefore have:

$$(3) \qquad \partial\phi / \partial x = \sum_{j=1}^{n} K\beta_j (x_j - x) \left[(x_j - x)^2 + (y_j - y)^2 \right]^{K/2-1} = 0$$

$$(4) \qquad \partial\phi / \partial x = \sum_{j=1}^{n} K\beta_j (y_j - y) \left[(x_j - x)^2 + (y_j - y)^2 \right]^{K/2-1} = 0$$

These equations cannot be solved explicitly for x and y. However, we can use an iterative approach similar to that employed in Cooper [1], [2], [3], and Kuhn and Kuenne [5]. This is derived from (3) as follows. Let us define, using (20) from the Appendix

$$G_j \equiv D_j^{K-2} \equiv \left[(x_j - x)^2 + (y_j - y)^2\right]^{(K-2)/2} \tag{5}$$

Then from (3) and (20) we have:

$$Kx \sum_{j=1}^{n} \beta_j G_j = K \sum_{j=1}^{n} \beta_j x_j G_j \tag{6}$$

$$x = \sum_{j=1}^{n} \beta_j x_j G_j \Big/ \sum_{j=1}^{n} \beta_j G_j \tag{7}$$

Similarly, from (4),

$$y = \sum_{j=1}^{n} \beta_j y_j G_j \Big/ \sum_{j=1}^{n} \beta_j G_j \tag{8}$$

Equations (7) and (8) can be used as iteration formulas for x and y in:

$$x^{(i+1)} = \sum_{j=1}^{n} \beta_j x_j G_j^{(i)} \Big/ \sum_{j=1}^{n} \beta_j G_j^{(i)} \tag{9}$$

$$y^{(i+1)} = \sum_{j=1}^{n} \beta_j y_j G_j^{(i)} \Big/ \sum_{j=1}^{n} \beta_j G_j^{(i)} \tag{10}$$

where the superscript i indicates the function evaluated at the i^{th} iteration, i.e.,

$$G_j^{(i)} = \left[(x_j - x^{(i)})^2 + (y_j - y^{(i)})^2\right]^{(K-2)/2}$$

In order to start the iterative process we have used as starting values the weighted mean co-ordinates:

$$x^{(0)} = \sum_{j=1}^{n} \beta_j x_j \Big/ \sum_{j=1}^{n} \beta_j \tag{11}$$

$$y^{(0)} = \sum_{j=1}^{n} \beta_j y_j \Big/ \sum_{j=1}^{n} \beta_j \tag{12}$$

Equations (9), (10), (11), (12) could also be employed to solve:

$$Min\ \phi = \sum_{j=1}^{n} \beta_j \left[(x_j - x)^2 + (y_j - y)^2\right]^{K/2}, \quad 0 < K < 1$$

However, by Theorem 2, we know that the iteration process, if it converged, might converge to a local optimum. The results of extensive calculations will be presented in Section 4.

3. SOME SIMPLE EXAMPLES OF THE EXTENDED PROBLEM.

As was stated in the introduction (Section 1), there are many significant, practical problems in which the cost of shipment (for example) from a source to many different sinks or destinations can more readily be represented in terms of a linear combination of weighted distances which have been raised to some power, i.e.,

$$\phi = ad^K \tag{13}$$

rather than

$$\phi = ad \tag{14}$$

where ϕ = cost, d = distance and a is a constant of proportionality. In Figure 1, the differences are contrasted and illustrate the results of Theorem 1.

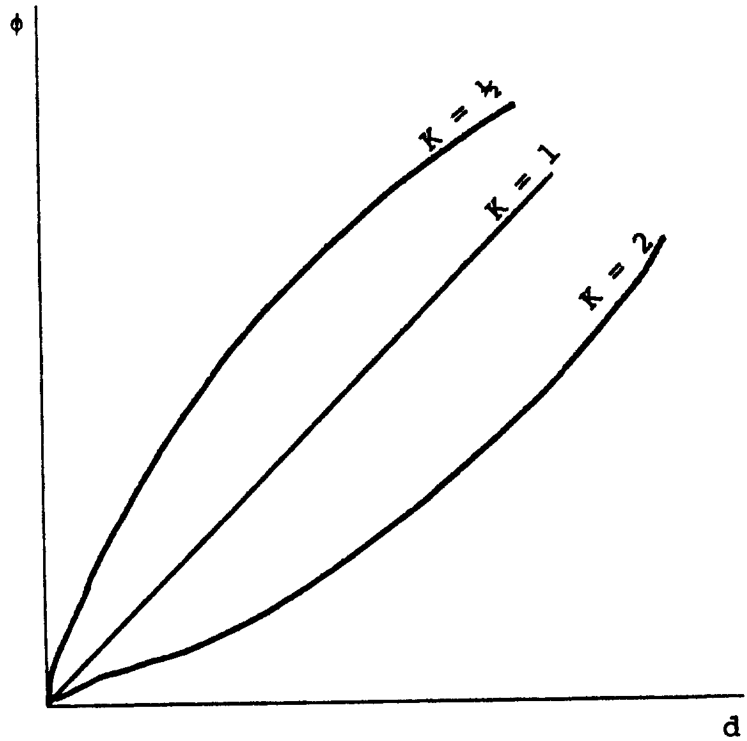

FIGURE 1: Powers of the Distance Variable.

It is readily seen from Figure 1, that when $K \geq 1$, as was proven in Theorem 1, the cost function ϕ is a convex function. Figure 1 also shows an illustration for $K = 1/2$ when ϕ is not convex. Since economies of scale are likely to lead to a cost function of the type $\phi = ad^k$ and $k < 1$, this is an important class of problems. Whatever computational methods can be devised for this case will have use. Unfortunately, this is the most difficult case to characterize in any simple fashion as Theorem 1 shows. However, some empirical computational results are given in Section 4.

As a specific illustration of how the value of K in Equation (2) affects the value of the solution vector $\bar{x} = (x,y)$ to (2), let us consider the following problem. Find the vector $\bar{x}$ to minimize ϕ when $(x_1, y_1) = (0, 0)$, $(x_2, y_2) = (1, 0)$, $(x_3, y_3) = (0, 1)$

when $K = 1/2$, $(x, y) = (0, 0)$
when $K = 1$, $(x, y) = (.2113, .2113)$
when $K = 2$, $(x, y) = (.3333, .3333)$

It can be seen from this example that more realism in the fitting of cost data can have a significant effect on the solution of a particular problem.

In Theorem 2, we showed that each destination point was a local minimum. In this example one of these destinations is also a global minimum. It turns out, however, that this is not always so. The global minimum can be taken on at points other than the destination points. Evidence for this assertion is given in the next section.

4. CALCULATION RESULTS

In Section 2 a method of iteration similar to that of Cooper [1], [2], [3], and Kuhn and Kuenne [5] was proposed to find the optimal location point (x, y). Extensive tests have been run using equations (9) and (10) as iteration equations and employing (11) and (12) as starting values. Table 1 gives the result of these calculations. The iteration technique converged easily for 180 problems. The destination sets were generated randomly and are available from the author on request.

Table 1 provides abundant evidence that the iteration technique of Cooper [1], [2], [3], and Kuhn and Kuenne [5] which was devised originally for the problem of Equation (1) will be just as effective for the extended problem

$$Min\ \phi = \sum_{j=1}^{n} \beta_j \left[(x_j - x)^2 + (y_j - y)^2\right]^{K/2}, \quad K \geq 1$$

For these problems, it was shown in Theorem 1 that every local minimum is a global minimum since we are minimizing a convex function over a closed convex set.

The iteration procedure that is given in Equations (9), (10), (11), (12) in no way depends upon the fact that $K \geq 1$. Hence we could also employ this procedure for $0 < K < 1$. However, as was shown in Theorem 1, the function to be minimized in this case is not necessarily convex. Hence, the value of ϕ to which the iteration procedure converges would not necessarily be the global minimum.

TABLE 1. Calculation Results $(K \geq 1)$

Destination Set No.*	Number of Destinations	K	ϕ min	x	y	Number of Iterations
1	10	1.00	303.261	28.814	49.053	27
		1.50	1866.70	29.486	53.537	9
		2.25	31509.6	31.657	55.680	6
		2.50	82459	32.604	56.201	9
		2.75	217212	33.592	56.678	15
		3.00	574938	34.573	57.119	27
2	10	1.00	341.936	60.445	30.371	15
		1.50	2171.89	59.902	35.715	7
		2.25	36265.2	61.224	40.924	6
		2.50	93411.1	61.593	42.112	9
		2.75	241328	61.901	43.112	16
		3.00	625103	62.180	43.966	35
3	10	1.00	377.706	34.820	38.192	14
		1.50	2411.86	37.530	36.082	6
		2.25	39920.7	40.562	34.932	5
		2.50	102324	41.304	34.735	7
		2.75	262969	41.946	34.592	12
		3.00	677531	42.506	34.491	21
4	20	1.00	793.065	52.178	52.318	20
		1.50	5315.95	47.654	51.003	8
		2.25	96244.1	45.634	49.835	5
		2.50	255069	45.271	49.476	8
		2.75	678456	44.975	49.124	11
		3.00	1810248	44.727	48.783	17
5	20	1.00	690.681	37.103	56.000	61
		1.50	4602.63	42.529	54.297	8
		2.25	81349.3	46.954	50.942	5
		2.50	213034	47.727	50.208	8
		2.75	559323	48.301	49.616	13
		3.00	1472151	48.726	49.138	20
6	20	1.00	786.478	54.396	50.724	12
		1.50	5106.66	52.802	52.094	6
		2.25	87289.0	51.467	53.035	4
		2.50	226597	51.216	53.218	6
		2.75	590295	51.029	53.360	9
		3.00	1542810	50.892	53.470	13
7	30	1.00	1127.93	36.036	43.008	54
		1.50	7380.62	39.602	46.604	8
		2.25	128569	40.732	48.799	5
		2.50	336520	40.960	49.218	8
		2.75	884521	41.170	49.555	12
		3.00	2333870	41.368	49.833	19
8	30	1.00	1179.02	42.277	50.488	12
		1.50	7778.77	43.348	51.149	6
		2.25	137143	44.528	50.806	4
		2.50	359926	44.864	50.586	7
		2.75	947959	45.176	50.362	11
		3.00	2504871	45.678	50.145	18

*Destination sets are available from the author on request.

TABLE 1. Calculation Results ($K \geq 1$) (Continued)

Destination Set No.*	Number of Destinations	K	ϕ min	x	y	Number of Iterations
9	30	1.00	1139.15	52.253	43.657	17
		1.50	7433.43	53.612	44.441	6
		2.25	128844	55.864	46.527	5
		2.50	335888	56.439	47.101	7
		2.75	878238	56.937	47.590	10
		3.00	2302463	57.367	48.026	16
10	40	1.00	1437.89	42.554	42.720	10
		1.50	9147.64	43.270	44.572	6
		2.25	154653	43.737	46.814	5
		2.50	401216	43.821	47.451	7
		2.75	1045348	43.882	48.037	9
		3.00	2733847	43.925	48.574	14
11	40	1.00	1594.14	51.517	58.597	14
		1.50	10565.8	51.131	55.837	7
		2.25	187212	50.619	53.843	5
		2.50	492098	50.478	53.438	7
		2.75	1298093	50.352	53.113	10
		3.00	3435436	50.237	52.850	14
12	40	1.00	1702.73	56.800	52.156	15
		1.50	11407.84	53.238	50.736	7
		2.25	201417	51.489	49.981	4
		2.50	526964	51.212	49.859	6
		2.75	1381802	51.012	49.771	9
		3.00	3631310	50.867	49.708	12
13	50	1.00	1927.11	47.826	59.905	7
		1.50	12629.0	47.608	59.950	6
		2.25	223068	47.912	59.349	4
		2.50	587620	48.011	59.068	6
		2.75	1555570	48.090	58.770	9
		3.00	4136649	48.143	58.462	13
14	50	1.00	1935.66	50.803	43.181	17
		1.50	12677.4	51.298	44.798	7
		2.25	218938	51.332	46.241	4
		2.50	569399	51.262	46.513	7
		2.75	1484741	51.183	46.718	10
		3.00	3881024	51.100	48.876	15
15	50	1.00	1895.34	47.247	54.381	15
		1.50	12356.9	47.976	53.226	6
		2.25	215199	48.592	52.239	4
		2.50	563227	48.706	52.034	6
		2.75	1480204	48.788	51.875	8
		3.00	3904956	48.846	51.753	11

*Destination sets are available from the author on request.

Table 2 presents a similar set of results as Table 1 except that $0 < K < 1$ for these calculations.

The results of the calculations presented in Table 2 indicate that, in general, the iterative method represented by Equations (9), (10), (11), (12) is no more difficult to use when $0 < K < 1$ than when $K \geq 1$. However, there is the problem of local minima to contend with. We know, from Theorem 2, that every destination is a local minimum. However, not every local minimum (and, therefore, the global minimum) need be at a destination. This will be demonstrated further on in this section.

To determine whether or not the local minima, to which the iterative method converged, were also global minima, a computer program was written to calculate the value of ϕ over a fine rectangular grid of points which included all the destination points. This made it possible to find all the local optima.

As two examples, consider destination sets 1 and 2 in Table 2. For all of these cases listed, the iterative calculation, starting always at the mean coordinates of the destination set (all $\beta_i = 1$ for calculations in Tables 1 and 2) arrived at the global optimum. This is all the more remarkable since the computational method uses the equations that represent necessary (but not necessarily sufficient) conditions for the existence of a local maximum or minimum. Yet for destination sets 1 and 2, it never found a local maximum and always found a global minimum. This behavior is not atypical. It occurs more often than not. Reasons for this will be discussed later.

For example, for destination set No. 4 when $K = 0.15$, the global minimum occurred at (69,61) and ϕ had a value of 32.7627. The iterative procedure found a local minimum at (57,51) and ϕ had a value of 33.0799. Both of these points are destination points. However, when $K = 0.35$, the iterative procedure found the global minimum at (57,51), a destination point. The value of ϕ was 68.8240. When $K = 0.50$ the iterative procedure converged to (57,51), the global minimum with $\phi = 120.299$. When $K = 0.75$, the procedure converged to (57,51) which is the global minimum. When $K = 0.85$, the procedure found (55.439, 52.068) which is the global minimum. When $K = 0.95$, the procedure found (53.122, 52.423) which is the global minimum. With destination set No. 6 and $K = 0.75$, the iterative procedure converged to (55.266, 49.105) which is not a destination point but is the global minimum. The value of ϕ at this point is 310.953. For this same destination set and $K = 0.85$, the iterative procedure converged to (54.949, 49.939), the global minimum with $\phi = 450.418$. This point is not a destination point. For $K = 0.95$, the procedure converged to (54.582, 50.500), the global minimum with $\phi = 652.99$.

There is no obvious *a priori* reason for this iterative calculation to converge most of the time to the global minimum, whether the global minimum is a destination point or not. In fact this seems very surprising, since when the global minimum is found it is in the presence of many "nearly" local minima.

From extensive perusal of computer output of the fine grid calculation of the value of the objective function, the reason for the above phenomenon becomes apparent. The local minima which are not global minima are very "shallow" or weak minima compared to the global minimum which is usually found by the iterative calculation. An example of a portion of the grid for a typical local minimum is as follows.

TABLE 2. Calculation Results ($0 < K < 1$)

Destination Set No.	Number of Destinations	K	ϕ min	x	y	Number of Iterations
1	10	0.15	15.256	32.00	44.000	16*
		0.35	30.058	32.00	44.000	10*
		0.50	50.885	32.00	44.000	12*
		0.75	123.93	32.00	44.000	30*
		0.85	177.50	29.577	46.049	33
		0.95	253.59	28.905	48.084	29
2	10	0.15	15.654	67.000	23.000	7*
		0.35	31.350	67.000	23.000	8*
		0.50	54.149	67.000	23.000	11*
		0.75	136.72	67.000	23.000	89*
		0.85	197.52	62.210	28.062	25
		0.95	284.70	60.848	29.691	17
3	10	0.15	15.927	27.000	53.000	10*
		0.35	33.352	27.000	53.000	14*
		0.50	58.833	27.000	53.000	25*
		0.75	150.40	33.856	41.978	36
		0.85	217.27	34.133	39.830	25
		0.95	314.07	34.566	38.618	17
4	20	0.15	33.080	57.000	51.000	7*
		0.35	68.824	57.000	51.000	8*
		0.50	120.30	57.000	51.000	10*
		0.75	307.99	57.000	51.000	20*
		0.85	449.86	55.439	52.086	37
		0.95	656.43	53.122	52.423	23
5	20	0.15	31.820	37.000	56.000	8*
		0.35	63.649	37.000	56.000	8*
		0.50	108.85	37.000	56.000	8*
		0.75	271.41	37.000	56.000	9*
		0.85	393.50	37.000	56.000	11*
		0.95	572.18	37.000	56.000	20*
6	20	0.15	33.076	60.000	32.000	17*
		0.35	69.650	52.000	30.000	40*
		0.50	123.37	55.191	35.810	68
		0.75	310.95	55.266	49.105	25
		0.85	450.42	54.949	49.939	18
		0.95	652.99	54.582	50.500	14
7	30	0.15	49.855	36.000	43.000	6*
		0.35	102.46	36.000	43.000	6*
		0.50	177.06	36.000	43.000	7*
		0.75	444.68	36.000	43.000	9*
		0.85	644.54	36.000	43.000	10*
		0.95	935.64	36.000	43.000	18*
8	30	0.15	50.361	37.000	41.000	17*
		0.35	106.08	37.959	47.380	99
		0.50	183.90	39.475	48.363	39
		0.75	463.71	41.407	49.617	20
		0.85	672.91	41.814	50.015	16
		0.95	977.70	42.136	50.347	13

*Solution point is a destination.

TABLE 2. Calculation Results ($0 < K < 1$) (Continued)

Destination Set No.	Number of Destinations	K	ϕ_{min}	x	y	Number of Iterations
9	30	0.15	50.121	53.000	46.000	11*
		0.35	103.54	53.000	46.000	6*
		0.50	179.38	53.000	46.000	7*
		0.75	450.75	53.000	46.000	24*
		0.85	652.75	52.146	44.008	30
		0.95	945.85	52.178	43.694	20
10	40	0.15	65.187	25.000	33.000	27*
		0.35	137.12	37.482	39.358	53
		0.50	234.37	40.125	40.296	33
		0.75	577.61	41.815	41.607	16
		0.85	830.95	42.169	42.073	12
		0.95	1197.2	42.440	42.510	11
11	40	0.15	67.594	44.000	61.000	15*
		0.35	139.82	58.000	73.000	19*
		0.50	244.72	58.000	73.000	29*
		0.75	625.08	51.972	60.966	25
		0.85	908.25	51.697	59.886	18
		0.95	1321.2	51.564	58.995	15
12	40	0.15	68.003	68.000	56.000	7*
		0.35	143.95	68.000	56.000	9*
		0.50	254.02	68.000	56.000	12*
		0.75	659.48	62.167	53.892	32
		0.85	963.79	59.302	53.025	22
		0.95	1408.4	57.488	52.407	17
13	50	0.15	84.493	53.000	51.000	12*
		0.35	174.89	53.000	53.000	25*
		0.50	303.93	51.429	58.123	33
		0.75	762.00	48.667	59.533	19
		0.85	1103.3	48.199	59.733	14
		0.95	1599.6	47.919	59.860	9
14	50	0.15	84.315	49.000	46.000	6*
		0.35	174.57	49.000	46.000	8*
		0.50	302.84	49.000	46.000	13*
		0.75	763.03	50.699	41.426	43
		0.85	1106.3	50.668	42.403	27
		0.95	1605.9	50.748	42.961	20
15	50	0.15	84.266	48.000	50.000	7*
		0.35	173.62	48.000	50.000	11*
		0.50	299.28	47.000	64.000	46*
		0.75	751.05	46.719	55.581	28
		0.85	1086.4	46.956	54.971	21
		0.95	1573.9	47.156	54.551	16

*Solution point is a destination.

$K = 0.85$

Destination Set No. 1	y 97.9999900000	98.0000000000	98.0000100000
11.9999900000	286.1615756351	286.1615909723	286.1616448381
12.0000000000	286.1615407302	$\boxed{286.1615190976}$	286.1616099332
12.0000100000	286.1615443539	286.1615596911	286.1616135568

(12, 98) is a local minimum.

The example below gives the neighborhood of the global minimum for this destination set.

$K = 0.85$

Destination Set No. 2	45.0	46.0	47.0
29.0	177.6723	177.5346	177.9097
30.0	177.5757	$\boxed{177.5179}$	177.6051
31.0	177.6417	177.7408	177.9097

It can be seen that the global minimum at (30, 46) is a much stronger minimum than the local minimum at (12, 98). Actually (30, 46) are not the correct co-ordinates of this point. The value found by the iterative calculation is (29.577, 46.049). However this minimum is sufficiently strong to appear on the grid as shown above at (30, 46). There were local minima not far from the global minimum, e.g., one at the destination point (32, 44).

Several perturbation studies were made to see how general this behavior is. What was done was to perturb the starting values from the usual starting values given by equations (11) and (12). In one case using Destination Set No. 1 with $K = 0.75$ where the global optimum occurred at (32, 44) and which had local optima at each of the destinations, such diverse starting values as (30.80, 55.10), (32.50, 44.10), (22.50, 71.90) all converged to (32, 44) when, for example, there is a local optimum at (22, 71).

This does not always occur, of course. However, it occurs sufficiently often to make this iterative calculation often produce the global optimum when $K < 1$. In any case, one can "reasonably" be sure of a global optimum by trying a series of widely scattered starting values. The convergence of the iterative calculation is very powerful, regardless of what it converges to. No case of divergence was ever observed. This agrees with experience with this method with $K = 1$. (See Cooper [1], [2], [3], and Kuhn and Kuenne [5].)

5. SUMMARY

An extension of the generalized Weber problem has been presented which should allow greater realism and greater ease of fitting cost data to be present in location studies in many areas of operations analysis. The nature of the solution has been characterized mathematically and a solution method has been presented. The results of extensive computation on a digital computer have also been presented.

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APPENDIX

Theorems 1 and 2 are proved below.

Theorem 1: The function

$$\phi = \sum_{j=1}^{n} \beta_j \left[(x_j - x)^2 + (y_j - y)^2\right]^{K/2}, \quad K > 0$$

is a convex function for $K > 1$. If $K < 1$, ϕ is neither convex nor concave.

Proof: A twice differentiable function $\phi\ (x, y)$ in an open convex set is convex if and only if

$$Q = (\partial^2\phi / \partial x^2)\lambda^2 + 2(\partial^2\phi / \partial x \partial y)\lambda u + (\partial^2\phi / \partial y^2) u^2$$

is positive semi-definite for any scalars λ, u. (See [4, p. 80].) We first compute some necessary quantities.

$$(15) \qquad \partial\phi/\partial x = -\sum_{j=1}^{n} K\beta_j (x_j - x) \left[(x_j - x)^2 + (y_j - y)^2\right]^{K/2-1}$$

$$(16) \qquad \partial\phi/\partial y = -\sum_{j=1}^{n} K\beta_j (y_j - y) \left[(x_j - x)^2 + (y_j - y)^2\right]^{K/2-1}$$

$$(17) \qquad \partial^2\phi/\partial x^2 = K(K-2) \sum_{j=1}^{n} \beta_j (x_j - x)^2 \left[(x_j - x)^2 + (y_j - y)^2\right]^{K/2-2}$$

$$+ K \sum_{j=1}^{n} \beta_j \left[(x_j - x)^2 + (y_j - y)^2\right]^{K/2-1}$$

(18) $$\partial^2\phi/\partial y^2 = K(K-2)\sum_{j=1}^{n}\beta_j (y_j-y)^2\left[(x_j-x)^2+(y_j-y)^2\right]^{K/2-2} + K\sum_{j=1}^{n}\beta_j\left[(x_j-x)^2+(y_j-y)^2\right]^{K/2-1}$$

(19) $$\partial^2\phi/\partial x\partial y = K(K-2)\sum_{j=1}^{n}\beta_j (x_j-x)(y_j-y)\left[(x_j-x)^2+(y_j-y)^2\right]^{K/2-2}$$

Let

(20) $$D_j \equiv \left[(x_j-x)^2+(y_j-y)^2\right]^{1/2}$$

We rewrite (17), (18) and (19) as:

(21) $$\partial^2\phi/\partial x^2 = K(K-2)\sum_{j=1}^{n}\beta_j (x_j-x)^2 D_j^{K-4} + K\sum_{j=1}^{n}\beta_j D_j^{K-2}$$

(22) $$\partial^2\phi/\partial y^2 = K(K-2)\sum_{j=1}^{n}\beta_j (y_j-y)^2 D_j^{K-4} + K\sum_{j=1}^{n}\beta_j D_j^{K-2}$$

(23) $$\partial^2\phi/\partial x\partial y = K(K-2)\sum_{j=1}^{n}\beta_j (x_j-x)(y_j-y) D_j^{K-4}$$

Using (21), (22) and (23) we evaluate Q:

$$Q = K(K-2)\lambda^2\sum_{j=1}^{n}\beta_j (x_j-x)^2 D_j^{K-4} + K\lambda^2\sum_{j=1}^{n}\beta_j D_j^{K-2}$$

$$+ 2K(K-2)\lambda\mu\sum_{j=1}^{n}\beta_j (x_j-x)(y_j-y) D_j^{K-4}$$

$$+ K(K-2)\mu^2\sum_{j=1}^{n}\beta_j (y_j-y)^2 D_j^{K-4} + K\mu^2\sum_{j=1}^{n}\beta_j D_j^{K-2}$$

(24)

$$Q = K(K-2)\sum_{j=1}^{n}\beta_j\left\{\left[(x_j-x)D_j^{(K-4)/2}\lambda\right] + \left[(y_j-y)D_j^{(K-4)/2}\mu\right]\right\}^2 + K\sum_{j=1}^{n}\beta_j D_j^{K-2}(\lambda^2+\mu^2)$$

It is obvious that since $\beta_j \geq 0$ and $K > 0$, then $Q \geq 0$ for $K \geq 2$.

As a first result we see that Q is positive semi-definite for $K \geq 2$ and hence ϕ is a convex function for $K \geq 2$. This result can be sharpened, however. Let us examine the quantity $B - |A|$ where, referring to (24) we have:

$$A \equiv K(K-2) \sum_{j=1}^{n} \beta_j \left\{ \left[(x_j - x) D_j^{(K-4)/2} \lambda \right] + \left[(y_j - y) D_j^{(K-4)/2} \mu \right] \right\}^2$$

$$B \equiv K \sum_{j=1}^{n} \beta_j D_j^{K-2} (\lambda^2 + \mu^2)$$

(25)
$$B - |A| = K \sum_{j=1}^{n} \beta_j \left[(x_j - x)^2 + (y_j - y)^2 \right]^{(K-2)/2} (\lambda^2 + \mu^2)$$

$$+ K(K-2) \sum_{j=1}^{n} \beta_j \left\{ (x_j - x) \left[(x_j - x)^2 + (y_j - y)^2 \right]^{(K-4)/4} \lambda \right.$$

$$\left. + (y_j - y) \left[(x_j - x)^2 + (y_j - y)^2 \right]^{(K-4)/4} \mu \right\}^2$$

Let $\left[(x_j - x)^2 + (y_j - y)^2 \right] \equiv S_j = D_j^2$, then

$$(26) \quad \frac{B - |A|}{K} = \sum_{j=1}^{n} \beta_j \left\{ S_j^{(K-2)/2} \lambda^2 + S_j^{(K-2)/2} \mu^2 \right.$$

$$+ (K-2) \left[(x_j - x)^2 S_j^{(K-4)/2} \lambda^2 + (y_j - y)^2 S_j^{(K-4)/2} \mu^2 \right.$$

$$\left. \left. + 2 (x_j - x) (y_j - y) S_j^{(K-4)/2} \lambda \mu \right] \right\}$$

which reduces to

$$(27) \quad \frac{B - |A|}{K} = \sum_{j=1}^{n} \beta_j S_j^{(K-4)/2} \left[(K-1) (a_j \lambda + b_j \mu)^2 + (a_j \mu - b_j \lambda)^2 \right]$$

where $a_j \equiv x_j - x$ and $b_j \equiv y_j - y$.

It is clear from (27) that if $K \geq 1$, then $(B - |A|) / K \geq 0$ and Q is positive semi-definite and hence, ϕ is a convex function for $K \geq 1$.

Let us now consider $K < 1$. Let

$$(28) \quad \frac{B-|A|}{K} = \sum_{j=1}^{n} \beta_j S_j^{(K-4)/2} F_j$$

where $F_j = (K-1)\ (a_j\lambda + b_j\mu)^2 + (a_j\mu - b_j\lambda)^2$

Let us examine each F_j when $K < 1$. It can readily be seen that by the choice of a suitable set of destination points and hence a suitable choice of a_j and b_j and therefore F_j we can determine

$\frac{B-|A|}{K} = \sum_{j=1}^{n} \beta_j S_j^{(K-4)/2} F_j$ to be either greater than zero or less than zero since λ, μ are completely arbitrary real numbers. Therefore, for $K < 1$, ϕ is neither convex nor concave.

Theorem 2: The points (x_j, y_j) $j = 1, \ldots, n$ are local minima in the problem:

$$Min \quad \phi = \sum_{j=1}^{n} \beta_j \left[(x_j - x)^2 + (y_j - y)^2 \right]^{K/2}, \quad 0 < K < 1$$

Proof: First, we may note that at the destination points (x_j, y_j) $j = 1, \ldots, n$, the first partial derivatives (components of the gradient of ϕ) are not defined. We see this as follows:

$$(29) \quad \partial\phi/\partial x = \sum_{j=1}^{n} \left[K\beta_j\ (x - x_j) \Big/ C_j^{(2-K)/K} \right]$$

$$(30) \quad \partial\phi/\partial y = \sum_{j=1}^{n} \left[K\beta_j\ (y - y_j) \Big/ C_j^{(2-K)/K} \right]$$

where

$$(31) \quad C_j\ (\bar{x}) = \left[(x_j - x)^2 + (y_j - y)^2 \right]^{K/2}$$

and $\quad \bar{x} = (x, y)$

It can be seen from (29), (30) and (31) that when $x = x_j$ and $y = y_j$, the derivatives become infinite.

We can abbreviate the above as follows.

$$\phi(\bar{x}) = \sum_{j=1}^{n} \beta_j [C_j(\bar{x})] \tag{32}$$

$$\partial\phi/\partial x = \sum_{j=1}^{n} \beta_j (\partial/\partial x)\, C_j(\bar{x}) \tag{33}$$

$$\partial\phi/\partial y = \sum_{j=1}^{n} \beta_j (\partial/\partial y)\, C_j(\bar{x}) \tag{34}$$

$$-\nabla\phi = \sum_{j=1}^{n} \beta_j \left\{K(\bar{x}_j - \bar{x}) \Big/ [C_j(\bar{x})]^{(2-K)/K}\right\} = -\sum_{j=1}^{n} \beta_j \nabla C_j(\bar{x}) \tag{35}$$

where $\bar{x}_j = (x_j, y_j)$

Equation (35) expresses the negative of the gradient vector of ϕ which is of interest in the determination of local minima. Since the derivatives are not defined at points (x_j, y_j) we can not use the usual necessary and sufficient conditions (which assume differentiability) to determine local minima.

Let $|\nabla C_j(\bar{x}_j)| = C_{ij}$ where $\bar{x}_i = (x_i, y_i)$ and define $C_i = \max_j C_{ij},\ j \neq i$.

Now let us choose a t_i-neighborhood of $\bar{x}_i$ such that

$$|\nabla G_j(\bar{x}_i + \bar{y})| \leq 2C_i \tag{36}$$

for $j \neq i$ and for any $\bar{y}$ with $|\bar{y}| < t_i$. Now let us consider a change from $\bar{x}_i$ to $\bar{x}_i + t\bar{z}$, where x_i is any destination point, and where $|\bar{z}| = 1$ and $\bar{z} = [z_1, z_2]$

$$(d/dt)\,\phi(\bar{x}_i + t\bar{z}) = \sum_{\substack{j=1 \\ j\neq i}}^{n} \beta_j (d/dt)\, C_j(\bar{x}_i + t\bar{z}) + \beta_i (d/dt)\, C_i(\bar{x}_i + t\bar{z}) \tag{37}$$

But from (36) it follows that:

$$(d/dt)\, C_j(\bar{x}_i + t\bar{z}) \geq -2C_i \tag{38}$$

From (37) and (38) it can be seen that:

$$(d/dt)\,\phi(\bar{x}_i + t\bar{z}) \geq -\sum_{\substack{j=1 \\ j\neq i}}^{n} 2\beta_j C_i + \beta_i (d/dt)\, C_i(\bar{x}_i + t\bar{z}) \tag{39}$$

Let us compute the second term of (39):

$$(40) \qquad C_i\ (\bar{x}_i + t\bar{z}) = \left[(x_i - x_i - tz_1)^2 + (y_i - y_i - tz_2)^2 \right]^{K/2}$$

$$= (t^2 z_1^2 + t^2 z_2^2)^{K/2} = t^K (z_1^2 + z_2^2)^{K/2} = t^K$$

therefore $\quad (d/dt)\ C_i\ (\bar{x}_i + t\bar{z}) = (z_1^2 + z_2^2)^{K/2}\ (K)\ t^{K-1} = Kt^{K-1}$

since $|\bar{z}| = 1$.

From (39) and (40) we have:

$$(41) \qquad (d/dt)\ \phi\ (\bar{x}_i + t\bar{z}) \geq -\sum_{\substack{j=1 \\ j \neq i}}^{n} 2\beta_j C_i + K\beta_i / t^{(1-K)}$$

Now, if we choose $t_1 < t_i$ such that $K\beta_i / t_1^{(1-K)} > \sum_{\substack{j=1 \\ j \neq i}}^{n} 2\beta_j C_i$ since $0 < K < 1$,

it is clear that in this t_1-neighborhood $(d/dt)\ \phi\ (\bar{x}_i + t\bar{z}) > 0$ for $0 < t < t_1$.

We therefore have a local minimum of $\phi(\bar{x}_i)$ for all destinations $\bar{x}_i$, $i = 1, \ldots, n$.