Reweighted range voting – new multiwinner voting method

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August 6, 2005

Abstract — **Reweighted range voting (RRV), a new multiwinner voting scheme, is defined and its main properties elucidated. Compared with good quality single transferable vote (STV) schemes, RRV is algorithmically simpler, grants voters greater expressivity (RRV votes are real vectors), is monotonic, and is less vulnerable to painful tied and near-tied elections. RRV also seems less "chaotic" than STV in its operation and seems to encourage voter honesty to a greater degree (although I have no precise definition of exactly what these things mean). RRV reduces to "range voting," the experimentally best 1-winner scheme, in the 1-winner case. Like good quality STV schemes and my previous "asset voting," RRV obeys a proportionality theorem and offers good immunity both to "candidate cloning" election-manipulation attempts, and to the "wasted vote" strategic problem that plagues 3-candidate plurality elections.**

We also reexamine some fundamental issues about multiwinner elections, such as "why is proportionality a good thing?" and "what is the chance an STV election will be nonmonotonic?" and we prove a new fundamental impossibility theorem.

1 Proportionate versus disproportionate representation

Many sources, ranging from the early thinkers [5] to more modern texts [8][9][6], took it for granted that "proportional representation" in parliaments is a good goal. Instead, we now genuinely examine the question.

If party A has 3 times as many votes as party B, does that mean it should get 3 times as many seats?

Perhaps you think minorities tend to be oppressed and hence deserve a little extra help, so A should only get 2 times as many seats. Or perhaps you think minorities are going to lose anyway, so it is better for society to accelerate their loss, so they should be granted a little less power, so that (say) A should get 4 times as many seats. These all initially seem defensible points of view.

So suppose you think getting X times as many votes entitles you to $F(X)$ times as many seats, for some increasing smooth function $F(X)$ with $F(0) = 0$ and $F(1) = 1$, but you are not sure what F should be. We can narrow down the choices:

$$F(a)F(b) = F(ab) \tag{1}$$

is required for self-consistency of your view. (This is by considering 3 parties P_1, P_2, P_3 where P_2 has a times as many votes as P_1 and P_3 has b times as many as P_2 and hence ab times as many as P_1.)

Fact 1 (All self-consistent kinds of representation). *The only self-consistent smooth monotonic $F(x)$ with $F(1) = 1$ and $F(0) = 0$ are: $F(x) = x^P$ where P is any constant positive power.*

If $P = 1$ we get proportional representation. But if $P \neq 1$, $P > 0$ then we get disproportionate representation: a fraction X of the population will get a number of seats proportional to X^P. Call this "P-power representation."

Fact 2 (Effect of chained elections on P). *If the population elects a subpopulation of representatives using P-power representation, then that subpopulation elects a subsubpopulation using Q-power representation, then the characteristic powers P in the two elections will multiply: the net effect is PQ-power representation.*

Now, why is it that, for the good of society, we should prefer $P = 1$? Suppose there is some yes/no question that needs to be decided. Presumably, the best attainable decision for the population as a whole would be reached if the entire population (after having studied the issue) voted yes or no on it.

Fact 3 (Subsampling and binary choices). *The same yes/no vote-percentages would happen (in expectation) if a random subsample of the population were selected (instead of the whole population),* they *studied the issue, and they voted on it.*

That is exactly the effect that is approximated by having a proportional representation parliament. But:

Theorem 4 (Subsampling and binary choices). *Any* dis*proportionate subsample of the population would in expectation vote with* different *percentages, on some binary issue B, than the whole population, and furthermore B may be chosen so that the two percentages are above and below 50%.*

Proof sketch: Make B be the following yes/no question: "If you are a type-X person, then say yes with probability p, otherwise say yes with probability q" where X is chosen so that type-X people are disproportionately represented in the sample, and the numerical values of p and q are chosen to cause vote percentages above and below 50% (since this is two linear equations in two variables, it is not hard to see there are

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always suitable p, q with $0 \le p, q \le 1$, indeed $p \approx q \approx 1/2$). Q.E.D.

Theorem 4 and fact 3 are a valid basis for claiming that proportional representation is better for society than disproportional representation.

Another advantage of proportional representation is this. Suppose there is a country which is divided into districts. If each district chose representatives via a truly proportionate scheme, with the number of representatives from each district truly proportional to its population, then (at least if we may ignore rounding-to-integer effects) the composition of the parliament could *not* be altered by evil attempts to redraw district boundaries. That would not be true if the districts employed disproportionate representation.

Many so-called "proportional" representation governments actually involve *cutoffs* in which, e.g., parties with below 5% of the vote get zero seats. Theorem 4 indicates that such cutoffs are a bad idea.

The USA's legislative bodies are elected via a system which makes little or no attempt to provide proportionate representation. As of August 2004, the total number of women in the US House is 62 out of 435 total members, and the total number of women in the 100-member Senate is 13. The total number of blacks in the US House is 38 and in the Senate is zero,[1] so that the black percentage (7%) of House and Senate members is approximately half their percentage (12.3%) in the US population as a whole. In the entire history of the USA, there have been only 5 black Senators. To my knowledge, no atheist has ever won a US House or Senate seat.

In 1929, the UK's Liberal party got 23.4% of the vote but less than 10% of the seats, as a result of the highly disproportional voting system used. This caused most voters to give up on the Liberal party as a probable "wasted vote" – so in the next election, it got only 7% of the votes. The net effect was, essentially, to destroy that party.

Moral: Proportional representation is a good goal, but one not achieved in 2004 USA or 1929 UK.

2 A note on notation

We will use a few notations which the less-mathematical among our readers may not know about. The *floor function* $\lfloor x \rfloor$ (pronounced "floor of x") is the greatest integer I with $I \le x$. For example $\lfloor 5.72 \rfloor = \lfloor 5 \rfloor = 5$. Similarly the "ceiling" function $\lceil x \rceil$ rounds *up*ward, so $\lceil 5.72 \rceil = \lceil 6 \rceil = 6$. A *$N$-vector* is an N-tuple of numbers. Variable names which are vectors are commonly written with a little arrow above them, in order to distinguish them from ordinary variables ("scalars"). For example $\vec{z} = (0, 5, 7)$ would be a 3-vector, while $q = 2$ would be a scalar. Vectors of the same size can be added: $(3, 4, 7) + (1, 0, 9) = (4, 4, 16)$. Vectors also may be multiplied by scalars, e.g. the product $q\vec{z}$ of the scalar q and the 3-vector $\vec{z}$ we just defined would be $q\vec{z} = (0, 10, 14)$.

3 STV, asset, and range voting

I believe the two most meritorious previously proposed multiwinner election methods are "Hare/Droop STV" (*s*ingle *t*ransferable *v*ote) and "asset voting." Actually these each really are a family of methods, comprising many variants of the same sort of idea, and I have no clear understanding of which variant is the best.

Short descriptions: Let there be N candidates from whom the voters must select W winners, $0 < W < N$.

Asset voting [11]: each vote is an N-vector of nonnegative reals whose entry-sum is 1. The ith coordinate of the sum of the vote-vectors is the amount of an "asset" that gets awarded to candidate i. The candidates then "negotiate"; any subset of the candidates can agree to redistribute the asset among themselves. After negotiations end, the W candidates with the most assets win. (Variants constrain the possible negotiations in various ways.)

STV: is too complicated to explain here in full detail. (A 20-line pseudocode procedure is given on page 7 of [11], and see ch.7 of [9] and [12].) But we shall explain it in the simplest case $W = 1$ where there is a *single* winner. Each vote is a preference ordering of the N candidates. The election proceeds in rounds. Each round the candidate top-ranked by the fewest votes is eliminated and erased from all preference orderings. We continue until only one remains; he is the winner. (Variants allow each preference ordering also to express indifference or ignorance, and there are various possible "quota" and "reweighting" schemes in the multiwinner case.)

Both these families have members which provably cause representation to obey some kind of "proportionality" [12][11] (under certain assumptions about constituencies of voters and candidates and how they will act). Specifically (the following theorem is stated more precisely, and proved, in [11]):

Theorem 5 (STV/Asset proportionate representation). *Suppose there are disjoint kinds of people. Specifically, in a V-voter, N-candidate, W-winner election, let the number of voters of type t be V_t, the number of candidates of type t be N_t, and the number of winners of type t be W_t. Further suppose that each type-t voter prefers each type-t candidate to every candidate of any other type, and says so in their Hare/Droop-STV vote. Define the "Droop Quota" Q by $Q = \lfloor V/(W+1) \rfloor + 1$. Then: $W_t \ge \lfloor V_t/Q \rfloor$ if $N_t \ge \lfloor V_t/Q \rfloor$. Under asset voting (if each type-t voter allocates all of his votes to type-t candidates and candidates of the same type agree to help each other), $W_t \ge \lfloor V_t(W + 1 - \epsilon)/V \rfloor$ where ϵ is any positive real, no matter how small, and provided $N_t \ge \lfloor V_t(W + 1 - \epsilon)/V \rfloor$.*

In other words, by running enough candidates and voting for them, each constituency[2] can guarantee getting at least a number of representatives essentially proportional to its membership.

Nevertheless, both STV and asset voting have disadvantages.

Range voting is experimentally the *best* 1-winner scheme for mapping votes to the identity of a single winner.[3] It is this:

[1] The confidences that these Senate black and female percentages are not mere statistical fluctuations are 99.9997% and $1 - 10^{-14}$ respectively.

[2] I use the word "constituency" in the same sense as Tideman [12] and the online WordIQ dictionary: "A constituency is any cohesive ... body [of people] bound by shared structures, goals or loyalty."

[3] My paper [10] is by far the largest, and perhaps the only, experimental and theoretical study of range voting in comparison with other voting systems. But I am not the inventor of range voting; it has been used in the Olympics.

each voter provides a real N-vector, each entry of which lies in the unit interval $[0, 1]$, as his vote (any other fixed range could be used instead). The vectors are summed and the index of the largest coordinate in the sum-vector is the winner.

So it is distressing that neither STV nor asset voting reduce to range voting in the 1-winner case.

Not coincidentally, STV is non-monotonic [3][4], i.e. giving a candidate your maximum possible vote can actually cause him to *lose*! Consequently STV voters can be motivated to express dishonest preferences in their votes – even in 3-candidate 1-winner elections – whereas [10] voters are never motivated to range vote in a manner whose $\le$-order relations are incompatible with their true opinion of the 3 candidates.

Another trouble with STV is that votes are preference orderings with no way for voters to describe the *intensities* of their preferences. In contrast, in both range and asset voting, the votes are vectors of continuously variable real numbers, allowing much more voter expressivity. Finally, asset voting is unconventional in that it is not a map from a set of votes to a set of winners at all – rather, the votes merely are used to provide variable amounts of an "asset" to each candidate, giving them more or less power in the later negotiation. Many people do not like this "negotiation" idea.

Our purpose in this paper is to present (in §5) a new multi-winner voting method without any of those disadvantages.

4 The probability of nonmonotonicity in STV 3-candidate 1-winner elections

STV elections can be nonmonotonic [3][4], i.e., voting for a candidate can actually cause him to lose. This is a major criticism of STV ([3]: "goes entirely against all principles of democracy") and, if this pathology is common enough, it is a major reason to replace it with the new system, RRV, that we shall describe in §5.

So how common is this? Crispin Allard [1] made an inexact geometric estimate of the probability of nonmonotonicity in a 3-candidate single-winner STV election and got 0.00025, suggesting this is not a problem in practice. However, his work had errors[4] We will now redo and correct Allard's estimate.

Let the three candidates be A, B and C.

Definition. A single-winner STV election will be said to be "nonmonotonic" if B wins, but if some subset of the voters, solely by changing their top-rankings *away* from A, can cause A to win.

The conditions for a STV monotonicity failure are:

1. A is ahead of B who is ahead of C in the first-place vote counts;
2. When C is eliminated, his transfers move B ahead of A so B is elected;
3. If some number of voters *switch* their top preference from A to C, so that both A and C are ahead of B, then when B is eliminated, A goes ahead of C, so that A is elected: nonmonotonicity!

Mathematically, these conditions are:

1. $a > b > c$.
2. $a < b + \alpha c$,
3. Some $x > 0$ exists so that $a - x > b$, $c + x > b$, and $a > c + 2x + \beta b$ where $\alpha = T_{CB} - T_{CA}$, $\beta = T_{BC} - T_{BA}$ where T_{ij} is the proportion of i's votes which transfer to j if i is eliminated.

Assuming $a > b > c$, by letting x slightly exceed $b - c$ we find that these conditions are equivalent to

$$a < b + \alpha c, \quad a + c > 2b, \quad a + c > (2 + \beta)b. \tag{2}$$

So we now can say the following.

Theorem 6 (Nonmonotonicity probability P). *Let P be the probability of nonmonotonicity in a 1-winner 3-candidate STV election, assuming that all vote fractions arise from a subdivision of the unit interval arising by placing an appropriate number of random-uniform points inside it. Then P is equal to the probability that if 7 real variables a, b, c, u, w, y, z are chosen uniformly from the following 4-dimensional subset of $\mathbb{R}^7$*

$$a + b + c = y + z = u + w = 1, \quad u, w, y, z > 0, \quad a > b > c > 0 \tag{3}$$

and $\beta = y - z$ and $\alpha = u - w$, that the conditions in EQ 2 all will be satisfied.

So far, we have simply intentionally followed Allard's reasoning, although we have cleaned it up somewhat and corrected Allard's mistake of forgetting to demand $a > b > c > 0$ in EQ 3, which made him off by a factor of 6. Aside from that, we both got the same result, which is that P is expressible as the 4-volume of a certain set. Now Allard attempted to estimate that volume and made errors while doing so. We instead take the simplest and most direct approach to computing volumes and to estimating probabilities: computer Monte Carlo integration[5] For those who know the computer language `C`, the core of the Monte Carlo program is the following:

```
Count=0;
for(i=0; i<N; i++){
  y = rand01(); z = 1.0-y;
  u = rand01(); w = 1.0-u;
  do{
    a=rand01(); b=rand01(); c=rand01();
    s = a+b+c;
  }while(s>1.0 || s==0.0);
  if(a<b){ t=a; a=b; b=t; }
  if(b<c){ t=b; b=c; c=t; }
  if(a<b){ t=a; a=b; b=t; }
  assert(a>b);  assert(b>c);
  a /= s; b /= s; c /= s;
  assert(a+b+c < 1.01);
  assert(a+b+c > 0.99);
```

[4] M.A.E.Dummett in his book [6] had already attacked Allard's estimate as likely to be erroneously too small, although he gave no evidence to back that accusation up.

[5] The computer program I used to do this may be downloaded from `http://math.temple.edu/~wds/homepage/works.html` #78. It might be possible for a more dedicated mathematician than I to compute an *exact* closed form for P by doing the integral.

```
  beta = y-z;
  alpha = u-w;
  if( a < b + alpha*c ){
    if( a+c > 2*b ){
      if( a+c > (2+beta)*b ){
        Count++;
  }}}
}
```

which performs `N` Monte Carlo experiments, of which `Count` are nonmonotonic, leading to the estimate $P \approx$ `Count/N`.

My 10^9 Monte Carlo experiments resulted in 10498423 monotonicity failures, which combined with a jackknife error estimate yields:

Fact 7. $P = (1.050 \pm 0.001)\%$.
Allard had underestimated P by two orders of magnitude. In STV elections with more than 3 candidates, nonmonotonicity can also happen in other ways and hence should be even *more* likely.

In other words, at least about one STV election in 95 would be nonmonotonic. There were 659 MPs in the UK House of Commons in 2001. So if the UK used STV single-winner (≥ 3)-candidate elections to choose them, we would expect 7 nonmonotonic elections. That means 7 very angry and fairly powerful robbed-winners, all demanding the blood of those fools who had advocated STV! Those who prefer to avoid that scenario are advised to advocate the (fully monotonic) RRV system proposed next section.

5 Reweighted range voting

Our new voting system, "reweighted range voting" (RRV), attempts to combine the advantages of range voting and Hare/Droop STV.

Let there be N candidates from whom V voters are to select W winners, $0 < W < N$, $0 < V$.

procedure Reweighted-Range-Vote
1: Each voter k supplies an N-vector $\vec{x}_k$ as his vote, each entry of which is a real number in $[0,1]$. The Cth entry of this vector expresses that voter's opinion of candidate C (i.e. 1=great, 0.5=middling, 0=terrible);
2: Each N-vector vote has associated with it, a "weight" $w_k \in [0,1]$.
3: **for** $r = 1$ to W **do**
4: **for** $k = 1$ to V **do**
5: Let the weight of vote k be $w_k = 1/(X+1)$, where the sum of vote $\vec{x}_k$'s winner-entries is X. (Thus, initially, there are no winners and all weights are 1.)
6: **end for**
7: Compute the weighted-vote-sum vector $\vec{s} = \sum_{k=1}^{V} w_k \vec{x}_k$ (actually, this step would be best programmed as combined into step 5, but we have written it separately to enhance clarity);
8: The candidate C with the largest $\vec{s}$-entry (among candidates who have not yet been declared "winners") is declared to be the rth winner.
9: **end for**

In the 1-winner case, RRV reduces to range voting [10], i.e., it simply adds up all the vote vectors $\vec{s} = \sum_{k=1}^{V} \vec{x}_k$ and then declares the winner to be the index of the largest entry in $\vec{s}$. The first RRV winner in fact is always the same as the range-voting winner, but the second RRV winner is not necessarily the same as the candidate range voting would say was in second-place. That is because the reweightings cause the supporters of the first winner to have diminished influence on the choice of the second.

Here is a list of **good properties of RRV:**

1. No "negotiation" is needed, unlike in asset voting [11].
2. RRV grants each voter greater freedom of expression than STV.
3. RRV is monotonic in the sense that top-ranking a candidate in your vote (or more generally simply increasing your vote for him) cannot hurt his chances of winning.
4. The weighting scheme seems to force fairly proportional representation. E.g. if 51% of voters vote (1,1,1,1,1,0,0,0,0,0) ("Republican") and 49% vote (0,0,0,0,0,1,1,1,1,1) ("Democrat") in a $N = 10$, $W = 5$ election, then the weightings will cause alternate elections of Republicans and Democrats.

   Here is another example, worked out by John Hodges (whom we quote):

   > 1000 fully-polarized voters, 10 seats, 20 candidates, two parties R and D, with 70% and 30% of the vote respectively. Then RRV gives the seats sequentially to R,R,D,R,R,D,R,R,tie, and whoever wins the tie loses the next one, so with ten seats the R's get 7 and the D's 3. Great.
   >
   > When W is small, how does this compare with the Droop Quota? With 2 seats, under Droop you would need 33% of the vote to get a seat, with 5 seats you would need 16.666%, with 8 seats 11.111%, so the above sequence of wins compares OK in fairness with the Droop Quota.

   We will in fact *prove* a proportionality theorem below.
5. Consequently there is considerable immunity to attempts to manipulate the election via candidate "cloning." (Manipulability by cloning is a well known deficiency of plurality voting.)
6. RRV has no "wasted vote" problem (also a well known defect of plurality voting) – at least in 3-candidate 1-winner elections. (Since range voting has no such problem: voting for an unlikely to win candidate C does not hurt any favorite – unless C actually does win.)
7. RRV's algorithmic complexities (both descriptive and computational), although substantially worse than range and asset voting, are simpler than most STV schemes.[6] That is especially true of STV schemes which allow voters to express *equalities* among candidates in their preference orderings or which allow the voters to only rank some, but not all, of the candidates.[7] It is actually rather unclear how STV systems should deal with

[6] A 20-line pseudocode procedure for STV voting is given on page 7 of [11]. The present RRV procedure is 9 lines.
[7] These changes would bloat STV's line count from 20 to more like 50.

voters who refuse to rank everybody. In RRV, equalities are trivial for voters to express, and voters could simply be allowed to complete their votes by saying "...and I award Z votes to each additional candidate" where Z is some number they specify ($0 \le Z \le 1$), which would make vote-completion easy.

8. RRV's vulnerability to painful near-tied elections (of the sort that keep plunging the USA into crisis) is not as bad as STV's since it does not have candidate "eliminations" (which lead to additional opportunities for vote-ties).
9. Eliminations in STV schemes have been criticized by Dummett [6] as capable of causing "chaotic" behavior – small changes in the input votes can get amplified to have large effects. RRV has no eliminations and all weight-changes are multiplications by factors< 1, i.e. the opposite of amplification. (These factors also are $\ge 1/2$ so that there are no giant decreases either; in the rth round all weights are between $1/r$ and 1 and hence can never get either extremely small or extremely large.) Hence RRV should largely "avoid chaos."
10. Finally, there is a certain amount of "encouragement of voter honesty" built in to RRV: you do not desire to exaggerate your opinion of some good candidate by too much, since that will (when he wins) decrease your vote-weight.

Theorem 8 (RRV proportionate representation). *Suppose there are disjoint kinds of people. Specifically, in a V-voter, N-candidate, W-winner election, let the number of voters of type t be V_t, the number of candidates of type t be N_t, and the number of winners of type t be W_t. Further suppose that each type-t voter awards each type-t candidate the maximum allowable vote* 1 *while giving each candidate of any other type* 0. *Then:*

$$|(W_t+1)V_s-(W_s+1)V_t| \le \min\{V_t, V_s\} \tag{4}$$

for each s, t with $V_t, V_s \ge 1$, provided enough type-s and type-t candidates are available so that we don't "run out of either prematurely." (It ought to suffice if $N_k \ge V_k W/V$ for each needed k.)

Proof: Let $W = V_a/V_b$ be the ratio of the number of type-a to type-b people. If J type-a candidates and K type-b candidates have been elected (so far), the weighted sum of the votes for any given as-yet-unelected type-a candidate will be $ZV/(J+1)$ whereas the the weighted sum of the votes for any given as-yet-unelected type-b candidate will be $V/(K+1)$. Thus if $J+1 > (K+1)Z$ then a type-b candidate will be elected before a type-a one, while if $J+1 < (K+1)Z$ then a type-a candidate will be elected first next. Therefore (assuming there are enough candidates of each type available, so that we do not run out, i.e. N_u is sufficiently large for each u of interest) RRV will produce W_t's such that $|(W_t+1)V_s/V_t-(W_s+1)| \le 1$ if $V_s \ge V_t \ge 1$. This proves the theorem. Q.E.D.

The way this is worded seems to be a stronger kind of statement about proportionality than in theorem 5. If anybody should desire it,[8] P-power *dis*proportionate representation would also be achieveable via `Reweighted-Range-Vote` by changing the weight formula in line 5 to $w_k = 1/(X+1)^P$.

6 A displeasing lack of symmetry?

The reader may have noticed that `Reweighted-Range-Vote` asymmetrically concentrates on winners and not losers. Similarly, Hare/Droop STV includes two different kinds of steps: those that select winners (who exceed the "Droop quota") and losers (who are "eliminated"; RRV somewhat resembles STV but has no eliminations). Thus both treat the two differently.

This seems peculiar. After all, a scheme for electing W winners from N candidates, may equally well be thought of as a scheme for choosing the $N-W$ *losers*. One might therefore imagine that "God's election method" instead would treat the two symmetrically. But in fact: doing so is *impossible*. We now prove this.

Desire #1 [proportional representation]: A fraction F of the voters (forming a "constituency") should be able to elect a fraction F of the seats.

Desire #2 [reversal symmetry]: Since choosing W winners from N candidates, can equally well be regarded as choosing the $N-W$ losers, we desire that if all votes are reversed=negated,[9] and the value of W changed to $N-W$, then the complement set should be elected.

Fact 9. *These two desires are incompatible.*

Remarks on the upcoming proof. One reader claimed the proof is wrong because "candidates have only one attribute." Yes, *if* we were just ordering candidates along a line, or just associating a single real number with each, then they would have only 1 attribute. However, we also are addressing the desire for proportional representation (PR). The whole concept of PR only has meaning at all (as in theorems 5 and 8) if there are parties or constituencies, and there can be many of these (in the following proof, there are 3 kinds). That is more than 1 attribute.

Now if we want to talk about both PR and reversal symmetry at the same time, then it would not be symmetric to consider only Republican-loved and Democrat-loved candidates. We need also to have Republican- and Democrat-*hated* ones. Then if one then tries to impose both PR and reversal symmetry, a contradiction results, proving fact 9.

Proof: Suppose 45% of Voters are Republican, 33% Democratic, and 22% Anarchist. Suppose candidates are either Republican-loved, Democrat-loved, or Anarchist-loved (disjointly from each other) and also are either Republican-hated, Democrat-hated, or Anarchist-hated (again disjointly from

[8] Dan Keshet also suggests changing the $1+X$ to $r+X$ where r is a positive constant. In particular he suggests $r = 1/2$. He argues that RRV with $r = 1$ is analogous to the "d'Hondt count" method of awarding seats to parties in party-list democracies. With $r = 1/2$ it instead would be analogous to the "pure Sainte-Lague" method: All parties initially have 0 seats; the party with largest quotient of votes divided by 0.5 more than its current number of seats, receives the next seat; this is repeated until the desired total number of seats has been awarded. Keshet's r-modified RRV still will obey proportionality theorem 4 except that the two occurrences of "+1" in EQ 4 both must be replaced by "$+r$."

[9] More precisely: to "reverse" preference ordering votes $A > B > C > D > \dots$ replace them with $\dots > D > C > B > A$. To "reverse" or negate a range vote involving numbers x in the range $[0, 1]$, replace them with $1-x$. (If the range instead were $[-1, 1]$ then we instead would negate $x \to -x$).

each other). Nobody is both loved and hated by the same person, but there are no further certainty-relationships, e.g. Anarchist-loved neither implies nor is implied by Democrat-Hated. There are thus exactly 6 possible kinds of candidates which we could denote[10] RD, RA, DR, DA, AR, AD.

Then: There simply is not, in general, any way to partition a given set of candidates into winners and losers with the winners being 45%, 33%, and 22% Republican-, Democratic-, and Anarchist-loved, respectively, and the losers being 45%, 33%, and 22% Republican-, Democratic-, and Anarchist-hated, respectively. Why? Because the following set of 12 simultanous linear equations in 12 variables

$$RA_w + RD_w = 45, \quad DA_w + DR_w = 33, \quad AR_w + AD_w = 22,$$

$$AR_\ell + DR_\ell = 45, \quad AD_\ell + RD_\ell = 33, \quad RA_\ell + DA_\ell = 22, \quad (5)$$

$$RA_w + RA_\ell = RA, \quad RD_w + RD_\ell = RD, \quad DA_w + DA_\ell = DA,$$

$$DR_w + DR_\ell = DR, \quad AD_w + AD_\ell = AD, \quad AR_w + AR_\ell = AR$$

(note: I have employed 2-letter variable names) then would have a solution no matter what 6 values RA, RD, DA, DR, AD, AR (summing to 100) were chosen for the right hand sides (as the percentages among the candidates of the 6 types of people). But this is a singular set of equations and hence generically has no solution. In particular, Gaussian elimination shows it has no solution when $RA = 14$, $RD = 15$, $DA = 16$, $DR = 17$, $AD = 19$, $AR = 19$. Q.E.D.

In light of this impossibility theorem, it is entirely proper for STV and RRV to concentrate asymmetrically on winners.

7 Conclusion

We began by re-examining some fundamental issues about proportional representation, finding some new justifications for it. (Actually, quite plausibly these realizations were not so much "new" as merely "not previously expressed in a formal manner.") We also found a simple, but profound, impossibility theorem about multiwinner voting schemes: reversal symmetry and proportional representation are incompatible desires.

We then quickly pointed out the most important advantages and defects of asset and STV voting, along the way correcting a wrong (but unfortunately widely quoted) numerical calculation by Allard which had made STV look too good.

Reweighted Range Voting (RRV) was then proposed as a new multiwinner voting method which is free of those defects.

We proved RRV obeys a proportionality theorem. All proportionality theorems in this paper are essentially of the form: if all voters are purely self-interested amoral racists, then they will be able to get the representation they deserve.

RRV is simpler to describe and use than STV, it is monotonic, and it allows voters much greater expressivity. The only clear advantage STV has over RRV is that in small elections carried out without computer aid, STV may be done by simply counting and sorting ballot papers repeatedly, with comparatively few real number arithmetic operations being required. That is because in a W-winner Hare/Droop-STV election, at most 2^{W-1} different weight values ever arise, because for each of $W-1$ winners one either applies, or does not apply, that winner's reweighting factor to each vote. If 2^{W-1} is substantially smaller than the number of voters, one can therefore substantially reduce the number of needed real arithmetic operations by sorting the ballot papers into 2^W different piles, each pile labeled with its weight value. (On the other hand some fancier STV schemes such as Meek's [7] actually require far *more* real arithmetic than RRV and definitely require a computer.)

So in my opinion these RRV advantages render STV obsolete in all but small elections done manually.

Asset versus RRV: Asset voting is simpler than RRV and arguably has certain further advantages (e.g.: candidates with too few votes to be elected still get *some* power, as seems "more fair"). But since "negotiation" is present in asset voting and not in RRV, the former is inapplicable if the entities being elected are not sentient. (It is left to the reader to determine whether that is the case in their election.) Thus both asset voting and RRV remain standing with neither obsoleted.

Take-home messages:

1. Proportional Representation is a good goal;
2. Our new RRV voting system is a good one that should be preferred to STV except perhaps in small manual elections;
3. It would be wise to recommend RRV and avoid recommending nonmonotonic systems such as STV, since nonmonotonicity occurs much more commonly than had been thought and would engender tremendous rage in its victims[11];
4. It is impossible to design a "reversal symmetric" PR voting system – explaining why neither STV nor RRV are reversal-symmetric.

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[10] The first letter in the 2-letter name says who loves you, the second who hates you.

[11] We quote [4]: "Most voters would probably be alienated and outraged upon hearing the hypothetical (but theoretically possible) election night report: 'Mr. O'Grady did not obtain a seat in today's election, but if 5000 of his supporters had voted for him in second place instead of first place, he would have won!'." The main factor saving STV from this fate is the fact [2] that it can be painful to recognize a nonmonotonic STV election, and perhaps, often, nobody made the effort. (This reference proved NP-completeness of the recognition problem, although not if the number of candidates is fixed.)

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[11] Warren D. Smith: "Asset voting" scheme for multiwinner elections, #77 at `http://math.temple.edu/~wds/homepage/works.html`.

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