Some Cool Formulas

Warren D. Smith*
wds@research.nj.nec.com

April 5, 1995

$$\binom{2x}{x} = \frac{4^x}{\sqrt{\pi x[1+J(x)]}}, \quad J(x) = \frac{2}{8x-1+} \frac{1\cdot 3}{8x+} \frac{3\cdot 5}{8x+} \frac{5\cdot 7}{8x+} \frac{7\cdot 9}{8x+} ..., \quad x > 0. \tag{1}$$

As you can see from the continued fraction above, S.A. Ramanujan (1887-1920) knew a thing or two. It was appreciated by T.S. Motzkin that continued fractions of the form

$$K(Y) = 4 + \frac{12}{15+Y-} \frac{3\cdot 5^2\cdot 7}{2\cdot 7^2+Y-} \frac{7\cdot 9^2\cdot 11}{2\cdot 11^2+Y-} \frac{11\cdot 13^2\cdot 15}{2\cdot 15^2+Y-} \frac{15\cdot 17^2\cdot 19}{2\cdot 19^2+Y-} ... \tag{2}$$

incorporate the most possible degrees of freedom for the least number of rational arithmetic operations into the approximation of the function of Y that they represent. With the $K(Y)$ above, Ramanujan's formula for $\binom{2x}{x}$ is

$$\binom{2x}{x} = \frac{4^x}{\sqrt{\pi(x + \frac{x}{xK(64x^2)-1/2})}}, \quad x > 0. \tag{3}$$

With G.H. Hardy (1877-1947), Ramanujan also played a role in the formula of H. Rademacher (1892-1969) for the number $p(n)$ of partitions of an integer n into integer parts. Thus $5 = 4+1 = 3+2 = 3+1+1 = 2+2+1 = 2+1+1+1 = 1+1+1+1+1$ so $p(5) = 7$. Of course,

$$\prod_{n\geq 1} \frac{1}{1-q^n} = \left\{ \sum_{n\in\mathbf{Z}} (-1)^n q^{n(3n+1)/2} \right\}^{-1} = \sum_{m\geq 0} q^m p(m). \tag{4}$$

But the real potato is

$$p(n) = \frac{1}{4\sqrt{3}(n-\frac{1}{24})} \sum_{k\geq 1} \frac{A_k(n)}{\sqrt{k}} \left[(1-\frac{k}{Q})\exp(\frac{Q}{k}) + (1+\frac{k}{Q})\exp(-\frac{Q}{k}) \right] \tag{5}$$

where $Q = \pi\sqrt{2/3}\sqrt{n-1/24}$ and

$$A_k(n) = \sum_{1\leq h\leq k,\ \mathrm{GCD}(h,k)=1} \cos(\pi[s(h,k) - \frac{2hn}{k}]) = O(k^{1/2+\epsilon}), \quad s(h,k) = \sum_{j=1}^{k-1} \frac{j}{k}((\frac{hj}{k})), \quad ((x)) = \left\{ \begin{array}{c} 0 \text{ if } 2x \text{ is an integer} \\ x - \lfloor x \rfloor - 1/2 \text{ otherwise} \end{array} \right\} \tag{6}$$

Thus

$$A_1(B) = 1, \quad A_2(B) = \cos(\pi B), \quad A_3(B) = 2\cos(\frac{\pi}{18} - \frac{2\pi}{3}B), \quad A_4(B) = 2\cos(\frac{\pi}{8} - \frac{\pi}{2}B) \tag{7}$$

$$A_5(B) = 2\cos(\frac{\pi}{5} - \frac{2\pi}{5}B) + 2\cos(\frac{4\pi}{5}B), \quad A_6(B) = 2\cos(\frac{5\pi}{18} - \frac{\pi}{3}B), \quad \text{and so on.} \tag{8}$$

And what is that "1/24" doing in (EQ 5)? Well... it is not coincidental that the "Leech Lattice" is 24 dimensional. It was shown by A.M. Odlyzko and N.J.A. Sloane that the maximum possible number of unit spheres that can touch one, in 24-dimensional space, is 196560, and this configuration is unique. John Leech discovered a 24-dimensional packing of equal spheres in which *every* sphere touches 196560 neighbors. The coordinates of the sphere centers are the integer linear combiantions of the following 24 row vectors:

*NECI, Princeton NJ

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( 8 | 0 0 0 0 0 0 0 0 0 0 0 | 0 0 0 0 0 0 0 0 0 0 0 | 0 )
( --+-----------------------+-----------------------+-- )
( 4 | 4 0 0 0 0 0 0 0 0 0 0 | 0 0 0 0 0 0 0 0 0 0 0 | 0 )
( 4 | 0 4 0 0 0 0 0 0 0 0 0 | 0 0 0 0 0 0 0 0 0 0 0 | 0 )
( 4 | 0 0 4 0 0 0 0 0 0 0 0 | 0 0 0 0 0 0 0 0 0 0 0 | 0 )
(...|          ...          |          ...          |...)
( 4 | 0 0 0 0 0 0 0 0 0 0 4 | 0 0 0 0 0 0 0 0 0 0 0 | 0 )
( --+-----------------------+-----------------------+-- )
( 2 | 2 2 0 2 2 2 0 0 0 2 0 | 2 0 0 0 0 0 0 0 0 0 0 | 0 )
( 2 | 0 2 2 0 2 2 2 0 0 0 2 | 0 2 0 0 0 0 0 0 0 0 0 | 0 )
( 2 | 2 0 2 2 0 2 2 2 0 0 0 | 0 0 2 0 0 0 0 0 0 0 0 | 0 )
(...|          ...          |          ...          |...)
( 2 | 2 0 2 2 2 0 0 0 2 0 2 | 0 0 0 0 0 0 0 0 0 0 2 | 0 )
( --+-----------------------+-----------------------+--
(-3 | 1 1 1 1 1 1 1 1 1 1 1 | 1 1 1 1 1 1 1 1 1 1 1 | 1 )
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And wouldn't you know it, of the 26 sporadic simple groups, 13 arise from the automorphism group of this lattice.

Speaking of spheres which touch, suppose you have $n+2$ spheres in n-space which all are pairwise tangent and whose radii are $1/k_j$, $j \in \{0,1,...,n+1\}$. Then T. Gosset (1869-1962) showed, extending an earlier result of R. Descartes (1596-1650) – this one is actually quite elementary, when you know about the properties of J. Steiner's (1796-1863) "inversive transformations" – that

$$n \sum_{j=0}^{n+1} k_j^2 = (\sum_{j=0}^{n+1} k_j)^2. \tag{9}$$

But arguably Bernhard Riemann (1826-1866) takes the cake for his formula for the number $\pi(x)$ of primes $\leq x$ (if x *is* a prime, use $[\pi(x-)+\pi(x+)]/2$). Let

$$f(x) = \sum_{n \geq 1} \frac{1}{n} \pi(x^{1/n}), \quad \pi(x) = \sum_{m \geq 1} \frac{\mu(m)}{m} f(x^{1/m}) \tag{10}$$

where $\mu(n)$ is the Moebius function, which is $(-1)^r$ if n has r prime factors and is squarefree, otherwise 0. Let

$$\mathrm{Li}(x) = \mathrm{PV} \int_0^x dx/\ln x = \mathrm{Ei}(\ln x), \quad \mathrm{Ei}(z) = \mathrm{PV} \int_{-\infty}^{z} \frac{e^t}{t} dt = \gamma + \ln z + \sum_{n \geq 1} \frac{z^n}{n!n}. \tag{11}$$

Then

$$f(x) = \mathrm{Li}(x) - \sum_{z,\ \zeta(z)=0,\ \mathrm{Re}(z)\neq 0} \mathrm{Ei}(z \ln x) + \int_x^\infty \frac{dt}{(t^2-1)t \ln t} - \ln 2. \tag{12}$$

Incidentally $\sum_{m \geq 1} \frac{\mu(m)}{m} \mathrm{Li}(x^{1/m}) = 1 + \sum_{n \geq 1} \frac{1}{n\zeta(n+1)} \frac{(\ln x)^n}{n!}$. The conjecture that the non-real zeros z of the Riemann zeta function (If $\mathrm{Re} s > 1$, then $\zeta(s) = \sum_{n \geq 1} n^{-s}$, otherwise use the analytic continuation) are all simple and lie on the line $\mathrm{Re} z = 1/2$ is the famous "Riemann hypothesis." A small calculation by te Riele and others has shown that it is true for the first 1.5×10^9 zeros.

My own contributions to the Formulaic Lore are perhaps not as impressive. You may be interested in my conjecture, which appeared in a paper with J.M. Smith (no relation) that, in 3-dimensional space, the "point set with the most shortcuts," (that is, for which the Steiner Minimal Tree is R times the length of the minimum spanning tree, and R is as small as possible) has

$$\begin{aligned} R &= \sqrt{\frac{283}{700} - \frac{3\sqrt{21}}{700} + \frac{9\sqrt{11-\sqrt{21}}\sqrt{2}}{140}} \\ &\approx 0.78419037337712224710839547781568775265 4. \end{aligned} \tag{13}$$

A rather peculiar number, don't you think? In 2-dimensional space, D-Z. Du and F.K. Hwang showed in 1992 that $R = \sqrt{3}/2 \approx .866025$, achieved by the vertices of an equilateral triangle. I call the point set that generalizes these both to n dimensions, the "n-dimensional sausage."

Consider an n-dimensional simplex whose edge lengths are e_{ij} for $i, j \in \{0,1,2,...,n\}$. The symmetric $(n+2) \times (n+2)$ matrix S of D.M.Y. Sommerville (1879-1934) has the e_{ij}^2 as its entries, but also an additional last row and column which are all 1's – except that all the diagonal entries of S are zero. Then, the n-volume V, the circumaradius R, and the inradius r of the simplex are given by

$$V = \frac{2^{-n/2}}{n!}\sqrt{\det S}, \quad R = \sqrt{\frac{1}{2}(S^{-1})_{n+1,n+1}}, \quad r = \frac{1}{\sqrt{2}\ \sum_{k=0}^{n} \sqrt{(S^{-1})_{k,k}}}. \tag{14}$$

The barycentric coordinates of the circumcenter are $u_i = -(S^{-1})_{d+1,i+1}$. If the altitude vectors are $\vec{h}_i$ and the face areas are F_i we have

$$\frac{1}{|\vec{h}_i|} = \frac{F_i}{nV} = \sqrt{2|(S^{-1})_{ii}|}. \tag{15}$$

The barycentric coordinates of the incenter are $u_i = \frac{r}{|\vec{h}_i|}$. The off-diagonal elements of S^{-1} may be used to find the areas of the 2-flats. Rather amazing how all this stuff may just be read off from the entries of the inverse of S, don't you think? This and many other such formulas were given in a paper I wrote which was *rejected* by *Linear Algebra and its Applications.* One consequence of these formulas is that the Delaunay triangulation of the BCC lattice, i.e. the tiling of space by the tetrahedra congruent to the one whose vertices are $\vec{0}$ and the permutations of $(1, 1, 0)$, is the one (of all known tilings of $\mathbf{E}^3$ by conguent simplices) whose simplices are the "roundest."

Let the volumes of the n-dimensional sphere of radius 1, and the n-dimensional regular simplex, octahedron, and cube of side 1, be respectively denoted by $\circ_n$, $\triangle_n$, $\diamond_n$, and $\square_n$. As everybody has known since the time of L. Schläfli (1814-1895),

$$\circ_n = \frac{\pi^{n/2}}{(n/2)!}, \quad \triangle_n = \frac{\sqrt{n+1}}{n!}2^{-n/2}, \quad \diamond_n = \frac{2^n}{n!}, \quad \square_n = 1. \tag{16}$$

Matters get more interesting in nonEuclidean space. I cannot give them here, but I have a great number of different formulas for the volumes and the dihedral angles of these objects in nonEuclidean n-space, and there are many suprising relationships among them. These should appear in a book to be written with R.H. Hardin and N.J.A. Sloane...