The Energy-Time Uncertainty Principle

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Abstract — **This paper contains a simple but powerful statement and proof of the Energy-Time uncertainty principle.**

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1 Introduction and motivation

My motivation to write this paper came from reading [20]. In this and other papers, Landauer argues that the energy time uncertainty principle does *not* force energy dissipation to occur during communication. Although Landauer's papers are interesting and important, and may be correct, they suffer from the major problem that Landauer nowhere *defines* exactly what he thinks the energy-time uncertainty principle is, nor does he provide a simple precise mathematical definition of "communication." It seems quite plausible to me that with varying, but still reasonable, definitions of these notions, one could reach varying conclusions on this fundamental question. Another motivation is the question of fundamental limits on computer speed: our (EQ 10) and its ilk[1] might be interpreted as saying that logical "states" of a computer can't change too quickly. For example, if "qubits" means "2-state systems," (EQ 13) tells us that a 90° "rotate" operation on one qubit within some multibit physical system requires time $\geq \pi\hbar/(\overline{E-E_0})$. where the expected energy of the physical system is $\overline{E-E_0}$ above the ground state.

After reading [20] I asked myself: just what *is* the energy time uncertainty principle, anyhow? After looking in about 20 quantum mechanics books, I was astonished to find (this is 1998!) that only one of them [28] managed to give an acceptable statement of it, where by "acceptable" I mean an inequality in which all variables have a mathematical definition. I then managed to produce such a statement myself, and found to my surprise that it seems to be substantially simpler and more powerful than previous ones.

Section 2 gives a history of uncertainty principles, including an amusing debunking of common myths about the Bohr-Einstein box debate. For another history quite complementary to ours, see [11]. Readers who just want the result may skip the historical section and go directly to section 3.

[1] Eq. 4 of [24] gives the same result as our (EQ 13), but weaker by a factor of 2.

2 History

Most quantum mechanics books state and derive the $\Delta p_j \Delta x_k \geq \hbar\delta_{jk}/2$ uncertainty principle, where Δp means the square root of the variance of the momentum p (as a random variable specified by a wavefunction) and Δx means the square root of the variance of the position x (ditto). Some go further and give a general uncertainty relation [27] among any two quantities corresponding to noncommuting operators[2]. Note: these relations are inequalities in which every term in the formula has a precise mathematical definition.

Almost all books mention the energy-time uncertainty principle, but apparently only Thirring's [28] actually gives an inequality in which every term in the formula has a mathematical definition. (He also gives a brief original proof.)

Theorem 1 (Thirring [28]) *Suppose Ψ is a wavefunction evolving with time t to $\exp(-i\hat{H}t/\hbar)\Psi$, where $\hat{H}$ is a Hamiltonian operator. Suppose the probability that an energy measurement on Ψ would yield a value in an energy interval of width ΔE, is $\geq 1-\delta$. Then*

$$\int_{-\infty}^{\infty} |\langle\Psi|e^{-iHt/\hbar}\Psi\rangle|^2 \mathrm{d}t \geq \frac{(1-\delta)^2}{h\,\Delta E}. \qquad (1)$$

But my theorem 3 seems to me to be less obscurely stated, easier to use, and has a more elementary proof. Also, the integral in Thirring's theorem often does not converge[3].

The energy-time principle is of a different character from the usual ones, because time does not correspond to a Hermitian operator with a complete set of orthonormal eigenstates. Energy (e.g. for systems with Coulombic, or more generally L^2+L^∞, interparticle potentials) does [15] [17]. Since time is just a real number, it commutes with everything.

There have been a few attempts in the literature[4] to formulate and prove energy-time uncertainty relations.

[2] Extensions of this include: other moments than 2 [10], and rather unclear claims about Shannon information [12]. Lieb [21] [22] has his own forms of the $\Delta p\Delta x$ uncertainty principle, based on Sobolev inequalities, which seem more useful for many purposes.

[3] A theorem very related to Thirring's (e.g. with the same left hand side) was given in [11].

[4] [18] gives a $\Delta E\Delta T$ uncertainty principle, where T is *not* time, but rather a new operator called "tempus." Finkel [10] can obtain a true $\Delta E\Delta t$ principle concerning "RMS state lifetime" but needs to introduce probabilities of state transitions and results about them from a different paper, while our result does not need to assume

Of these, the only ultrasimple proof besides mine and Thirring's [28] (at least, in the corrected form below) is Mandelstam and Tamm's [23]: From [27] we have

$$\Delta R\,\Delta S \;\geq\; |\overline{RS - SR}|/2 \tag{2}$$

where ΔR means the square root of the variance of R, and overline denotes expectation value, for any two operator observables R, S. The Ehrenfest expectation value version of Von Neumann's version of the Schrödinger time evolution equation is

$$\hbar\,\dot{\overline{R}} \;=\; -i\,(\overline{HR - RH}) \tag{3}$$

where H is the Hamiltonian. Hence

$$\Delta R\,\Delta H \;\geq\; \frac{\hbar}{2}|\dot{\overline{R}}|, \tag{4}$$

so that after time evolution for a time lapse t,

$$t(\Delta H)_{\max} \;\geq\; \frac{(\overline{R_{\rm new} - R_{\rm old}})\,\hbar}{2\,(\Delta R)_{\max}} \tag{5}$$

where the subscript "max" indicates the maximum value of the indicated standard deviation occurring throughout the time evolution of duration t. Hence

Theorem 2 (Corrected version of Mandelstam and Tamm [23]) *If we let $(\Delta t)_R$ denote the minimum amount of time required for the expectation value of R to change additively by $\pm(\Delta R)_{\max}$, i.e. by 1 of its own standard deviations, then*

$$(\Delta t)_R\,(\Delta H)_{\max} \;\geq\; \frac{\hbar}{2}. \tag{6}$$

The Mandelstam-Tamm theorem has the advantage that it works for *any* R, but it seems hard to use because $(\Delta R)_{\max}$ and $(\Delta H)_{\max}$ are rather strange quantities.

Among the most amusing historical developments in the early history of quantum mechanics were the oft-recounted debates between A.Einstein and N.Bohr[5]. The legends associated with these debates are positively permeated with the same aura as the famous story[6] of George Washington and the Cherry Tree.

Einstein at the 1930 Solvay conference [4] [5] presented to Bohr the following attack on quantum mechanics via the energy time uncertainty principle. Keep in mind that in 1930, it was still 15 years before the first precise statement of this principle had been made [23], so Bohr and Einstein were arguing about an issue neither understood. (They both thought the uncertainty principle was $\Delta E\Delta t \geq \hbar/2$ and with the interpretations of the Δ's the same as in the $\Delta p\Delta x$ principle. This interpretation has never been justified, and also note that the coefficient 1/2 on the right hand side is also unjustified, at least in theorem 3 and in [11].) Einstein considered a box being weighed by a spring scale and containing photons. A cuckoo-clock mechanism mounted on the box opens a door on the box's side for a time duration t, specified extremely precisely, and with a precision apparently independent of any quantity having to do with photon energies. The weight of the box before and after the measurement seems to be measurable extremely precisely, and hence, since $E = mc^2$, the energy of the escaping photons is deducible extremely precisely. Hence we get $\Delta E\Delta t$ (where ΔE and Δt denote uncertainties in the values of t and of the energy E of the photons that escaped through the door) arbitrarily small, "refuting" quantum mechanics. Of course, of the three formulations of the energy-time principle given here, and Finkel's [10], none of them pertain to "uncertainty in the time lapse" at all, so this entire idea of Einstein's was based on a false premise. But neither Einstein nor Bohr were aware of that, and so both thought this was a tremendously dangerous attack on the foundations of quantum mechanics. So then (the usual tale proceeds) Bohr after great effort refuted Einstein's attack by invoking general relativity (!), as follows. The uncertainty Δz in the vertical position z of the spring scale obeys $\Delta z\Delta p_z \geq \hbar/2$. Einstein presumably wants $\Delta p_z \ll g m_{\rm escaped\ photons} t$, where g is the acceleration of gravity. But now, the clock mounted on the box, according to general relativity, differs by Δt from the time on a clock attached rigidly to the Earth, where $\Delta t \approx gt\Delta z/c^2$. Hence our knowledge of the time lapse will suffer from an uncertainty of order $h/(m_{\rm escaped\ photons}c^2)$, saving the uncertainty principle $\Delta E\Delta t \gtrsim h$ and quantum mechanics. The fairy tale[7] concludes: Einstein was so stunned at this use of his own theory of general relativity against him, that he conceded defeat.

Actually, Bohr's counterattack, although startling, had two fatal flaws. First, I do not see what Δp_z has to do

any transitions (although it certainly still applies if there are any). Non-holonomy in Lie algebras is used by [14] in an extremely non-elementary paper without a clear statement of a physically meaningful result. Wigner's paper [30] seems to give also an exceedingly vague statement about "the spread in time of presence at a particular plane," whatever that is. Papers analyzing specific physical scenarios and which therefore cannot claim any universality include [7] and [25], although at least the latter is clearly written.

[5] For example, it's often recounted how Einstein, Podolsky, and Rosen mounted an attack [9] on quantum mechanics, which fortunately Bohr refuted [6]. This tale is nonsense, because Bohr's "refutation" is completely muddled, and if you or I had submitted it, would have been ripe for referee ridicule and rapid rejection. E, P, and R's reasoning is in comparison a masterpiece of clarity, and in fact the idea of "local hidden variable theories" their paper spawned (after additional clarification of what that might mean by later authors) was quite logically reasonable. Only much later was the full picture revealed. With the work of Kochen and Specker [19] it was seen that it was logically impossible for local hidden variable theories always to give the same predictions as quantum mechanics. Then work on Bell's inequality [8] developed to the point where an experimental test could be, and was, carried out [2] [3], with the result supporting quantum mechanics and refuting local hidden variable theories.

[6] Entirely fictional: this fabrication created by Parson M.W.Weems [29] in 1800 has forever after been inflicted upon US school children in an effort to make them realize the importance of honesty.

[7] For a typical rendition, see [26]. Omnes concludes that of course Bohr was right and in fact regards the proof by Thirring as a later justification, with the caveat that, presumably due to a weakness in his mathematical argument, Thirring did not manage to get the right coefficient 1/2 of $\hbar$. Well, no: Thirring's result does not assert what Omnes thought – it says nothing about uncertainty in a time measurement. Also, the tightness of the numerical coefficients in theorems 2 and 3 indicates that 1/2 is not "right," at least in these formulations.

with $gm_{\text{escaped photons}}t$. We could agree only to perform weighings after exponentially damping the box's motion by placing it in a bath of viscous oil; any uncertainties in the weighing would then seem to be of a fixed magnitude exponentially independent of anything else. (The box would be in a minimum uncertainty state after the damping, roughly.) Second, neither gravitational time dilation nor Δz need have anything to do with the case. This is because Einstein could instead have measured the mass of the box with an electric field (after charging the box with a known amount of charge). The capacitor plates generating the electric field could have been superconductively shorted during the door-cycling (so that the box would be exposed to precisely zero field during the door-cycling, avoiding all possibility of time dilation caused by Δz during that epoch) but charged to 1 volt for the weighings – say by connection to an arbitrarily enormous external capacitor bank providing a 1 volt reference. Thus, both steps in Bohr's argumentation depended on an extremely specific scenario (and were dubious even there) and don't impact on even slightly altered scenarios, much less the whole of physics.

Although the fact Bohr's argument was unjustifiable does not prove Einstein was right, I think he probably was – an opinion backed up by [1]. But whether he was or not, since no claim has been made that quantum mechanics asserts what Bohr and Einstein thought it asserted, this whole dialogue is irrelevant.

3 Result, proof, and remarks

Theorem 3 (Main result) *Let $0 \le \delta < 1$. Suppose a wavefunction Ψ has the property that the probability that an energy measurement would yield an answer lying inside an energy interval of width ΔE, is $\ge 1-\delta$. Suppose the wavefunction evolves in time according to Schrödinger's equation with some time-independent Hamiltonian for a time Δt, resulting in a time-evolved wavefunction Φ. Then*

$$|\langle\Phi|\Psi\rangle| \;\ge\; (1-\delta)\,\cos\frac{\Delta t\,\Delta E}{2\hbar}. \tag{7}$$

Proof. WLOG we assume Ψ is a linear combination of orthonormal energy eigenstates with coefficients c_E. Thus if I is the ΔE wide energy interval, the assumption is that

$$\sum_{E\in I} |c_E|^2 \;\ge\; 1-\delta. \tag{8}$$

Under Schrödinger time evolution, each energy eigenstate evolves in time by multiplication of c_E by the phase factor $\exp(-iEt/\hbar)$. The maximum difference in angle between any phase factor in I and the phase factor corresponding to the midpoint of I, is then $(\Delta E)t/(2\hbar)$. Assume for the purpose of getting a lower bound on correlation (since this is the worst that could possibly happen) that the phase angle for any eigenstate outside I gets rotated fully 90° away – but this corresponds to only a fraction $\le \delta$ of the probability. Then the inner product obeys (EQ 7), QED.

Remarks.

(i) In other words, if you want the time-evolved wave function to differ substantially from the original one ("differ" if the sense of having correlation smaller than some constant, i.e. "angle" between the two wavefunctions as Hilbert space vectors larger than some constant) you need to wait for a time Δt at least of order $\hbar/\Delta E$. Application to, e.g. state transitions or radiative decay is immediate.

(ii) Let $X = \overline{E-E_0} = \langle\Psi|H-E_0|\Psi\rangle$ be the expectation value of the energy above the ground state energy E_0. (The theorem that a ground state exists for any system with Coulombic potentials, was surveyed in [21]; see also lemma 1 of [16] combined with lemma 4 of Kato's [15] and its improvement proposition 1 of [17] for L^2+L^∞ interparticle potentials.) Then for any $k>1$, the probability that the energy lies in an interval $[E_0, E_0+Xk]$, is $\ge 1-1/k$ by Markov's inequality ([13] 26.1.36). Hence an immediate corollary is that

$$|\langle\Phi|\Psi\rangle| \;\ge\; (1-k^{-1})\,\cos\frac{\Delta t\,k\,(\overline{E-E_0})}{2\hbar}, \quad \text{for any } k>1. \tag{9}$$

In other words, it is impossible for a wavefunction to evolve to a state sufficiently different that it is at (say) a $\ge 45°$ angle away, unless the time lapse exceeds

$$\Delta t_{45°} \;\ge\; 0.1947\hbar \,/\, (\overline{E-E_0}). \tag{10}$$

More generally to evolve an angle θ away, one must wait

$$\Delta t_\theta \;\ge\; \frac{2\hbar}{k\,(\overline{E-E_0})}\,\arccos\frac{\cos\theta}{1-k^{-1}}, \quad 1<k\le\frac{1}{1-\cos\theta} \tag{11}$$

so that in particular (by considering $k=1/(1-\cos\theta)$)

$$\Delta t_\theta \;\ge\; \frac{2\hbar}{\overline{E-E_0}}(1-\cos\theta); \tag{12}$$

(EQ 10) came from using $\theta = 45°$ in (EQ 11) with the optimal value $k\approx 4.95646$.

(iii) The case $\theta = 90°$ is especially simple (we take the limit $k\to 1+$): to evolve to an orthogonal state requires waiting at least a time

$$\Delta t_{90°} \;\ge\; \pi\hbar \,/\, (\overline{E-E_0}). \tag{13}$$

(iv) We can also get a statement[8] about energy *variance* by applying Chebyshev's inequality ([13] 26.1.37) to see

$$|\langle\Phi|\Psi\rangle| \;\ge\; (1-k^{-2})\,\cos\frac{k\,\Delta t\,\Delta E}{\hbar}, \quad \text{for any } k\ge 1, \tag{14}$$

where now ΔE denotes the square root of the variance of E. Thus (by considering $k\to 1+$) evolving to an orthogonal state requires time

$$\Delta t_{90°} \;\ge\; \frac{\hbar}{\Delta E}; \tag{15}$$

to evolve to a state at least angle θ away requires

$$\Delta t_\theta \;\ge\; \frac{\hbar}{k\,\Delta E}\,\arccos\frac{\cos\theta}{1-k^{-2}}, \quad 1<k\le\frac{1}{\sqrt{1-\cos\theta}}. \tag{16}$$

[8] In contrast: In [25]'s final step he converts an explicit statement about an energy probability distribution into a one about a quantity "ΔE" which is not clearly defined, but which cannot be square root of variance, since the variance for his distribution is infinite.

Thus (by taking the optimal value $k \approx 2.522006$), evolving to a 45°-different state requires time

$$\Delta t_{45^\circ} \geq 0.2281 \frac{\hbar}{\Delta E}. \quad (17)$$

Similarly one could get results depending on any moment of E besides the 2nd.

(v) Inequality (EQ 7) is tight: consider two energy eigenmodes at the interval endpoints, each with amplitude $1/\sqrt{2}$.

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