Zaslavsky's Theorem

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Abstract

This paper is a retelling of the proof of Zaslavsky's Theorem. For any arrangement of hyperplanes, there is a corresponding semi-lattice detailing the intersections of the arrangement. The Möbius function assigns a specific value to each node of the semi-lattice. Zaslavsky's Theorem states that the addition of the absolute values for all of these Möbius function values is equal to the number of regions the corresponding hyperplanes split the vector space into. The paper will explain all necessary terminology, prove the theorem, and then show interesting results that can be derived from the theorem.

1 The Problem

Definition 1 **[2]** *Let* $\mathbf{K}$ *be a field and let* $V_{\mathbf{K}}$ *be a vector space of dimension* ℓ. *A* **hyperplane** H *in* $V_{\mathbf{K}}$ *is an affine subspace of dimension* $(\ell - 1)$. *An* **arrangement** $\mathcal{A}_{\mathbf{K}}$ *is a finite set of hyperplanes in* $V_{\mathbf{K}}$.

For example, if $V_{\mathbf{K}} = \Re^2$, then the hyperplanes will be one-dimensional lines. If the intersection of all the hyperplanes is nonempty we call the arrangement *central*. Otherwise we call the arrangement *affine*, or simply noncentral. In the interest of clarity, sometimes we refer to arrangements as hyperplane arrangements, or arrangements of hyperplanes.

Here are two examples of hyperplanes that will be used throughout the introduction:

*Written under the advisement of Curtis Greene. Written on LaTeX.

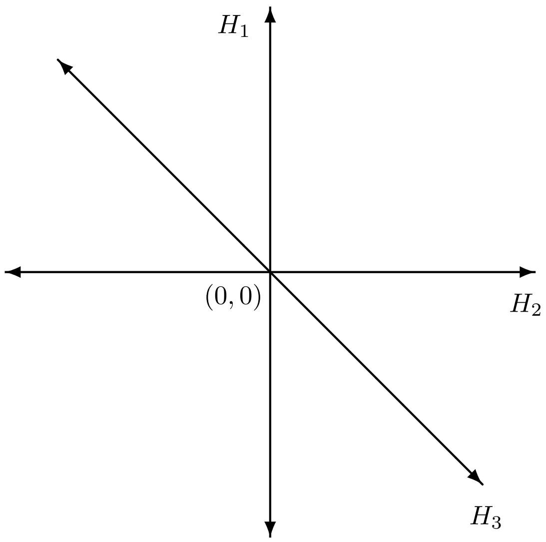


Figure 1: A central arrangement. The hyperplanes are labelled H_1, H_2, and H_3, which intersect at the origin.

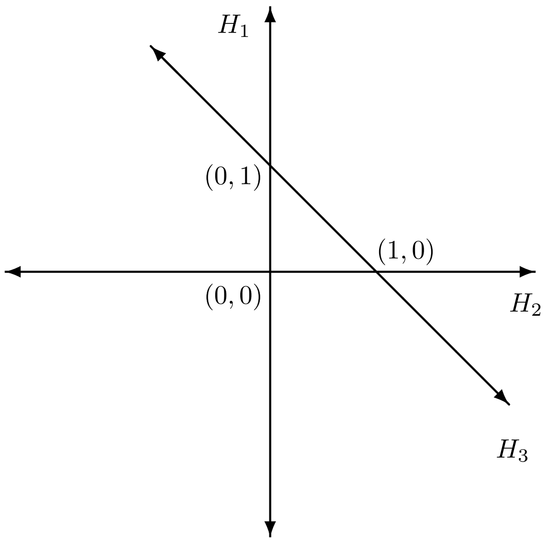


Figure 2: An affine arrangement

Definition 2 *For hyperplanes lying in* $\Re^d$, *the* **regions** *of a hyperplane arrangement* $\mathcal{A}$ *are the connected components of* $\Re^d - \bigcup_{\mathcal{A}} H$ *(the space minus the hyperplanes).*

We can think of regions as obtained by cutting along the hyperplanes, or taking the hyperplanes away, and taking note of the connected components. We say $N(\mathcal{A})$ is the number of connected components of arrangement $\mathcal{A}$.

The reader can see that Figure 1 splits the plane into six 2-dimensional regions, while Figure 2 splits it into seven regions. The problem is to find an easy way to count the number of regions into which an arrangement splits the vector space. When working with arrangements involving more hyperplanes and especially in higher dimensions, it may not be so easy to count the number of regions. Zaslavsky's Theorem solves this problem by using an alternate way to describe arrangements.

2 Introduction

2.1 Lattices

We will define lattices, and then describe how to associate a lattice with an arrangement. First, we need to state the fundamental properties of a partially ordered set (poset).

A poset consists of a set X and an order relation $\leq$ on X, which must satisfy the following properties:

- $x \leq x,\ \forall x \in X$
- $x \leq y,\ y \leq x \Rightarrow x = y,\ \forall x, y \in X$
- $x \leq y,\ y \leq z \Rightarrow x \leq z,\ \forall x, y, z \in X$

Remark: A partial order is an order defined for some, but not necessarily all pairs of items. For example, the sets $\{a, b\}$ and $\{a, c, d\}$ are subsets of $\{a, b, c, d\}$, but neither is a subset of the other. So "subset" is a partial order on sets. (Taken from [3]).

Definition 3 *A lattice is a partially ordered set with a join ($\vee$) operation and a meet ($\wedge$) operation. The join of two elements x and y is the least upper bound of x and y. The meet of two elements x and y is the greatest lower bound of x and y.*

The join and meet operations can be extended to arbitrarily finite collections (for example $x_1 \vee x_2 \vee \ \ldots\ \vee x_n$). A finite lattice must have a smallest element and a largest element.

Definition 4 *A $\wedge$-semi-lattice is a poset with a meet operation $x \wedge y$ for all pairs of elements x, y.*

A $\wedge$-semi-lattice may not have a join for some elements. Thus a finite $\wedge$-semi-lattice must have a smallest element, but may not have a largest element. A $\vee$-semi-lattice would be defined similarly.

We will now describe how a semi-lattice $L(\mathcal{A})$ can be associated with a hyperplane arrangement $\mathcal{A}$. To do this, we define $L(\mathcal{A})$ to be the set of all nonempty intersections of sets of hyperplanes in $\mathcal{A}$.

For example, in Figure 1 (three lines intersecting at a common point), there are 5 such intersections. The intersections are H_1, H_2, H_3, $H_1 \cap H_2 = H_1 \cap H_3 = H_2 \cap H_3 = H_1 \cap H_2 \cap H_3 = (0,0)$, and in addition, we list the empty intersection (the intersection of no hyperplanes), which is equal to $\Re^2$. So the set of intersections is $\{\ \Re^2\ , H_1, H_2, H_3, (0,0)\ \}$. The reader can check that the set L corresponding to the intersections for Figure 2 contains 7 elements.

2.2 Drawing a Diagram

Each element of L, that is, each intersection of hyperplanes of the arrangement, is a node of the diagram. The lower the dimension of the intersection, the higher the node in the diagram. Finally, a node is connected to another node if it shares a space in common. For example, if a point is the result of the intersection of the two lines h_1 and h_2, the node for that point will be higher than and connected to h_1 and h_2. Redundant lines are not drawn (such as the line from $\Re^2$ to $(0,0)$ in the figure below). Using our examples, which we show again here, we will label the points what they represent in the graph.

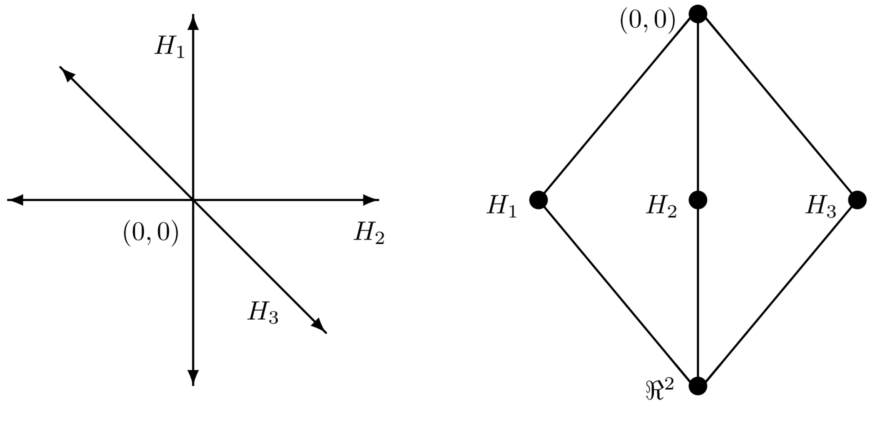


Figure 3: A central arrangement and its corresponding lattice.

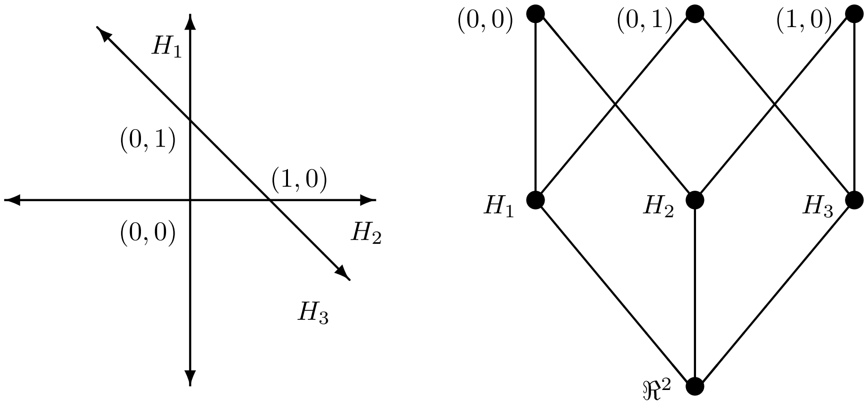


Figure 4: An affine arrangement and its corresponding semi-lattice.

2.3 More About Lattices

We define a partial order on L by:

$$X \leq Y \Leftrightarrow Y \subseteq X,$$

where X and Y are elements of L. This is reverse inclusion [2]. The whole space (in our examples $\Re^2$) is the minimal element. One note to remember is that while our examples in this paper take place in $\Re^2$ for convenience and understanding, we could be looking at hyperplanes in any dimension, such as planes in $\Re^3$, 3-spaces in $\Re^4$, etc.

Now we have our set L, and a partial order on L. Observe that L has the following properties (partially taken from [2]):

- L has a rank function given by $r(X) = codim X$. Thus $r(V) = 0$ and $r(H) = 1$ for $H \in \mathcal{A}$.
- The join $X \vee Y$ is given by $X \cap Y$, if it is nonempty.
- The meet $X \wedge Y$ is given by $\bigcap\{Z \mid Z \in L,\ X \cup Y \subset Z\}$.

We say that an element X covers an element Y if X has rank 1 greater than Y (that is, $r(X) = r(Y) + 1$). An **atom** is an element covering $\hat{0}$, the element with rank 0. Therefore, the atoms of $L(\mathcal{A})$ represent the hyperplanes $H \in \mathcal{A}$.

We observe that any two elements x, y in $L(\mathcal{A})$ have a meet, since the union of any two elements is always contained in the whole vector space V. However, two elements may not have a join. Consider the affine arrangement presented in Figure 2. Hyperplanes H_1 and H_2, intersect at a point, while hyperplanes H_1 and H_3 intersect at a different point. The two points do not coincide, so there is no element that corresponds to $H_1 \vee H_2 \vee H_3$. Thus, they have no join. Figure 1, however, has a join for any two elements, as the reader can check.

If the arrangement is central, $X \cap Y \neq \emptyset$ for all $X, Y \in L$, and thus $L(\mathcal{A})$ is a lattice, with the origin as the top element. If it is affine, it defines a semi-lattice with a bottom element but no top element.

In fact, $L(\mathcal{A})$ is always a *geometric* (semi-)lattice. A geometric (semi-)lattice has the following extra properties:

(i) Every element is a join of atoms.
(ii) If X covers $X \wedge Y$, then $X \vee Y$ covers Y, (if $X \vee Y$ exits).

The first property is obvious from the definition of L: since the atoms represent the hyperplanes, any intersection noted on the (semi-)lattice will have come from an intersection of hyperplanes, which is a join of the atoms by definition.

The second property follows because of the relationship between rank and dimension. Below, we prove the second property for central arrangements, which are geometric lattices, then extend the proof for affine arrangements, which yield geometric $\wedge$-semi-lattices. Recall that $X \wedge Y = \bigcap\{Z \mid Z \in L,\ X \cup Y \subset Z\}$. Also, recall from linear algebra that $X + Y = \{x + y | x \in X, y \in Y\}$. We have the following key facts:

- $X + Y$ is a subspace containing X and Y.
- If W is a subspace containing X and Y, then $W \supseteq (X + Y)$

Since $X \wedge Y$ is a subspace containing X and Y, we conclude that $(X \wedge Y) \supseteq (X + Y)$. Observe that $X + Y$ may not be a subspace contained in our lattice. It will, however, have codimension greater than or equal to $X \wedge Y$, since $(X \wedge Y) \supseteq (X + Y)$.

¿From linear algebra we know that:

$$dim(X) + dim(Y) = dim(X \cap Y) + dim(X + Y).$$

This means that the following also holds:

$$codim(X) + codim(Y) = codim(X \cap Y) + codim(X + Y).$$

Now we substitute $X \wedge Y$ for $X + Y$ to get:

$$codim(X) + codim(Y) \geq codim(X \cap Y) + codim(X \wedge Y).$$

Based on our definitions of meet and join, we now express this inequality in terms of rank:

$$r(X) + r(Y) \geq r(X \vee Y) + r(X \wedge Y).$$

Rearranging, we get:

$$r(X \vee Y) - r(Y) \leq r(X) - r(X \wedge Y).$$

If X covers $X \wedge Y$, the right side of the inequality equals 1. Looking at the left side of the inequality, $r(X \vee Y)$ is certainly not less than $r(Y)$ by definition. The ranks could also not be equal, since that would imply that X was less than Y, meaning $X = X \wedge Y$. So the left side of the inequality must equal 1 also, proving the result that $X \vee Y$ covers Y. $\square$

For affine arrangements, as long as $X \vee Y$ exists, we can ignore the hyperplanes and intersections that do not contain $X \cap Y$. Ignoring these elements gives a central arrangement (with $X \vee Y$ as the top element), and the proof above holds. Since we can do this for any X, Y in the $\wedge$-semi-lattice, the properties still hold for all elements. Therefore, in the affine case this proves that $L(\mathcal{A})$ is a geometric semi-lattice.

2.4 The Möbius Function

We now define the Möbius function, a function that attaches numerical values to the nodes in our diagrams. This next definition is taken from [1]:

Definition 5 *The Möbius function, denoted $\mu(x, y)$ is the unique function on $L \times L$ such that $\mu(x, y) = 0$ for $x \not\leq y$ and:*

$$\sum_{\substack{z \\ x \leq z \leq y}} \mu(x, z) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x < y \end{cases} .$$

We can compute $\mu(x, y)$ recursively, starting with $\mu(x, x) = 1$:

$$\mu(x, y) = - \sum_{\substack{z \\ x \leq z < y}} \mu(x, z) \tag{1}$$

as long as x, y, $z \in L$ and $x < y$, otherwise the Möbius function equals zero (for a similar way to define the Möbius function, refer to [2]).

Now we demonstrate using our two diagrams representing intersection (semi-)lattices of arrangements. The boxes show the values $\mu(x, y)$, where x is the bottom element. Notice these key facts:

- The bottom element always has the value one.
- The atoms (nodes of rank 1) always have the value -1. This corresponds to the fact that the only element less than an atom is the bottom element.

- The value of an element on the next row up is computed by adding together the values of the elements less than it, and taking the negative of that number.

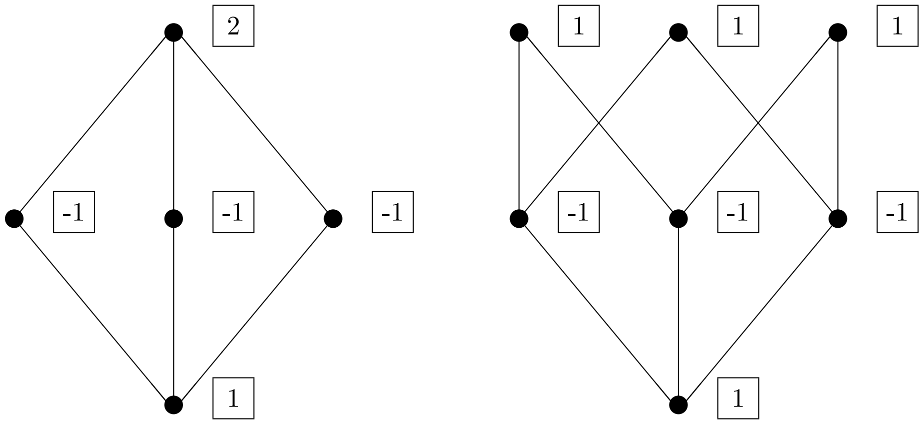


Figure 5: The Möbius function values of the nodes of the diagrams for our examples.

We have now introduced the tools for the solution to our region counting problem.

3 Useful Theorems and Definitions

The theorems and definitions in this section are somewhat technical, and while they are needed in the proof of Zaslavsky's Theorem, which we prove in the next section, to give the details there would be a distraction from the main ideas of the proof. Therefore, we present them in this separate section, so that they can be referred to in the main proof.

3.1 The Möbius Inversion Formula

Theorem 1 (Equivalent Möbius function definitions) *The Möbius function, which we defined as:*

$$\sum_{\substack{z \\ x \leq z \leq y}} \mu(x,z) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x < y \end{cases} \tag{2}$$

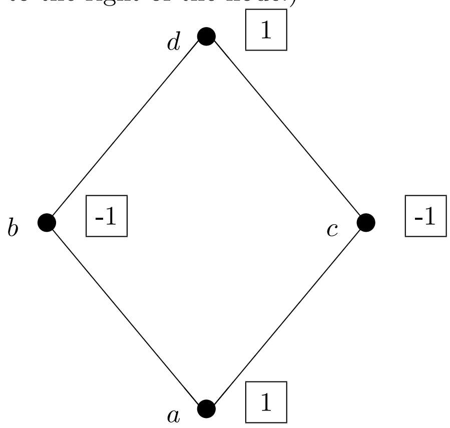
$$\text{and } \mu(x,y) = 0 \text{ if } x \not\leq y$$

may also be defined by:

$$\sum_{\substack{z \\ x \le z \le y}} \mu(z,y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x < y \end{cases} \tag{3}$$

$$\textit{and } \mu(x,y) = 0 \textit{ if } x \not\le y \; .$$

Proof:
In order to prove this theorem, we present an example illustrating the Möbius function values as entries in a matrix indexed by the nodes of a lattice. Here is an easy example, where the matrix computes $\mu(x,y)$ where x is the row value, and y is the column value, and the letters correspond to the nodes of the lattice shown (the nodes are lettered a through d on the left, with $\mu(a,x)$ put in a box to the right of the node.)



$$\begin{array}{cc} & y \\ & \begin{array}{cccc} a & b & c & d \end{array} \\ x \;\; \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{pmatrix} 1 & -1 & -1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

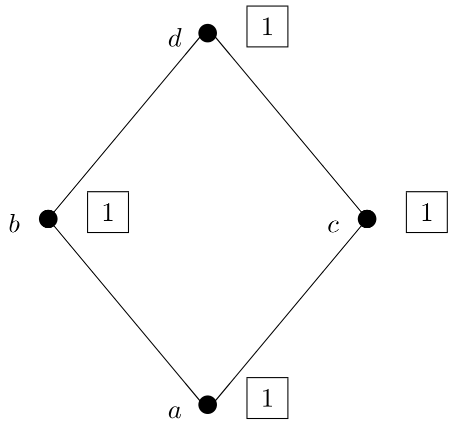
Figure 6: A lattice and its matrix of Möbius function entries.

The reader can check that the first row of the matrix corresponds to the Möbius function computed starting at the first node (the boxed values), the second column of the matrix corresponds to the Möbius function computed starting at the second node, and so on. This is the first instance we have encountered where neither x nor y in $\mu(x,y)$ have been constant.

Now we define a matrix $\zeta = (\zeta(x,y))_{x,y \in P}$ where

$$\zeta(x,y) = \begin{cases} 1 & \text{if } x \le y \\ 0 & \text{otherwise} \end{cases} \; .$$

Observe that ζ will be a matrix of 1s and 0s, and will be upper triangular. Using our lattice example, ζ would be:



$$
\begin{array}{cc}
 & y \\
 & \begin{array}{cccc} a & b & c & d \end{array} \\
x \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\end{array}
$$

Figure 7: A lattice and its matrix of ζ Function entries.

The proof of Theorem 1 is now a consequence of the following lemma:

Lemma 1

$$\mu\zeta = \zeta\mu = I.$$

Proof:
Say the lattice has j nodes (a through j where j is arbitrary). So μ and ζ are both $j \times j$ matrices. Look at $(\mu\zeta)_{xy}$, the value at row x, column y of the matrix of $\mu\zeta$. This is equal to $\sum_{z=a}^{j} \mu(x,z)\zeta(z,y)$ (the x-row of μ times the y-column of ζ).

All values where $z > y$ disappear, since then $\zeta(z,y) = 0$. When $z \leq y$, $\zeta(z,y) = 1$, so we now have $\sum_{\{z|z\leq y\}} \mu(x,z)$. When $x \not\leq z$, $\mu(x,z) = 0$, so now what is left is

$$\sum_{\substack{z \\ x \leq z \leq y}} \mu(x,z).$$

We now have our definition of the Möbius Function, which we know equals 1 if $x = y$ and 0 if $x < y$, and the whole sum equals 0 if $x > y$ (since there are no z's to sum). So the only time $(\mu\zeta)_{xy}$ is nonzero is when $x = y$, where the value is 1. This defines the identity matrix.

Now that we know $\mu\zeta = I$, we also know that the matrix multiplication $\zeta\mu$ will also yield the identity matrix. This is a simple (but deep) fact from linear algebra. Since ζ is a square matrix satisfying $\mu\zeta = I$, then $\zeta = \mu^{-1}$, and we have $\zeta\mu = I$. $\square$

To complete the proof of Theorem 1, we observe that $(\zeta\mu)_{xy}$ equals $\sum_{z=a}^{j} \zeta(x,z)\mu(z,y)$. When $x > z$, $\zeta(x,z) = 0$ and in all other cases it is 1. So the sum is equal to $\sum_{\{z|z \geq x\}} \mu(z,y)$. But $\mu(z,y) = 0$ when $z \nleq y$, so we have:

$$\sum_{\substack{z \\ x \leq z \leq y}} \mu(z,y),$$

which must equal 1 if $x = y$ and 0 otherwise (since $(\zeta\mu)_{xy}$ is a value of the identity matrix). We already proved that the object above must be the very same Möbius function since it is the unique inverse of ζ. So we now have another way to define the Möbius function. $\square$

Next we state the Möbius Inversion Formula, an important formula used throughout the proof of Zaslavsky's Theorem, and also in this section, to prove Weisner's Theorem. This is a combination of the Möbius Inversion Formula presented in [1] and in [2].

Corollary 1 (Möbius Inversion Formula) *Let f, g be functions on the lattice L with values in an abelian group. Then*

$$f(x) = \sum_{y \geq x} g(y) \iff g(x) = \sum_{y \geq x} \mu(x,y) f(y) \tag{4}$$

$$f(x) = \sum_{y \leq x} g(y) \iff g(x) = \sum_{y \leq x} \mu(y,x) f(y). \tag{5}$$

Proof:
The proof is elementary given the equivalent definitions of the Möbius Function in Theorem 1. Starting with statement (4) on the left, we envision f and g as column vectors. We notice that $\sum_{y \geq x} g(y) = f(x)$ is equivalent to $[\zeta]\vec{g} = \vec{f}$, because all terms with $y < x$ in $[\zeta]$ are 0, while the terms where $y \geq x$ in $[\zeta]$ are 1, so they will be mulitiplied by the corresponding value in the column vector $\vec{g}$ when doing matrix multiplication.

Starting from the fact that $[\zeta]\vec{g} = \vec{f}$, we proceed quickly:

$$[\zeta]\vec{g} = \vec{f} \iff [\mu][\zeta]\vec{g} = [\mu]\vec{f} \iff \vec{g} = [\mu]\vec{f}.$$

Writing out $\vec{g} = [\mu]\vec{f}$ gives us $g(x) = \sum_{y \geq x} \mu(x,y) f(y)$, which proves statement (4) completely. Statement (5) is proved similarly, using $\vec{g}^T[\zeta]$. $\square$

3.2 Weisner's Theorem

The next definition states the rules for a certain operation on posets to be a closure operator.

Definition 6 *A closure operator is a function $x \to \overline{x}$ such that*

$$\begin{array}{rc} (i) & x \leq \overline{x} \\ (ii) & x \leq y \Rightarrow \overline{x} \leq \overline{y} \\ (iii) & \overline{\overline{x}} = \overline{x} \end{array}$$

For example, if p is a fixed element of a lattice L, then the map $\phi : x \longrightarrow x \vee p$ is a closure operator on L. The proof is elementary, and is left as an exercise.

Now we have all the tools needed to prove Weisner's Theorem, an important theorem utilized in the proof of Zaslavsky's Theorem. Here we solve an important theorem for which Weisner's Theorem is a consequence of.

Consider the map $\phi : x \longrightarrow x \vee p$ for any element $x \in L$ and for some fixed $p \in L$. Let $\bar{L} \subseteq L$ denote the image of ϕ (that is, the elements y for which $x \vee p = y$ for some $x \in L$). Observe the fact that $\bar{L}$ consists of all y such that $y \geq p$.

Theorem 2 *For all $z \in \bar{L}$,*

$$\sum_{\substack{x \in L \\ x \vee p = z}} \mu(0, x) = \begin{cases} 0 & \textit{if } p > 0 \\ \mu(0, z) & \textit{if } p = 0 \end{cases} . \tag{6}$$

Proof:

For $z \in \bar{\mathrm{L}}$, define

$$g(z) = \sum_{\substack{x \in L \\ x \vee p = z}} \mu(0, x) \quad \textit{and} \quad f(z) = \sum_{\substack{y \in \bar{L} \\ y \leq z}} g(y).$$

¿From these definitions, we can see that $f(z)$ can be rewritten as

$$f(z) = \sum_{\substack{y \in \bar{L} \\ y \leq z}} \sum_{\substack{x \in L \\ x \vee p = y}} \mu(0, x).$$

By examining the elements taken in each sum, we can simplify this as

$$f(z) = \sum_{\substack{x \in L \\ x \leq z}} \mu(0, x).$$

So $f(z)$ sums $\mu(0,x)$ over the interval $[0,z]$ in L. Therefore, by the definition of the Möbius function, $f(z)=0$ if if $z>0$ and $f(z)=1$ if $z=0$. Now, we solve for $g(z)$ using the Möbius Inversion Formula, and get

$$g(z) = \sum_{\substack{y \in \bar{L} \\ y \leq z}} \mu(y,z)f(y) = \begin{cases} 0 & \text{if } p > 0 \\ \mu(0,z) & \text{if } p = 0 \end{cases} .$$

This last statement is true because if $p>0$, then $f(y)=0$ since every y is greater than 0 in L.

If $p=0$, then the only term that matters is the $y=0$ term, since all the other terms involve elements y that are greater than 0, and the argument above holds. So $f(0)=1$, and we are left with the one term $\mu(0,z)$.

This proves statement (6) for all $z \in \bar{L}$, and the theorem is proved. □

Corollary 1 (Weisner) *Let L be a finite lattice, and let $a \in L$, $a > 0$. Then, $\forall\, b \geq a$,*

$$\mu(0,b) = -\sum_{\substack{x < b \\ x \vee a = b}} \mu(0,x). \tag{7}$$

Proof: Suppose $a \in L$, $a > 0$. Then, by the above theorem, for all $b \geq a$,

$$\sum_{\substack{x \leq b \\ x \vee a = b}} \mu(0,x) = 0,$$

since $a > 0$ (since it is equivalent to our case of $p > 0$).

We can break this up into two cases, where either $x = b$ or $x < b$, and therefore we just get

$$\mu(0,b) + \sum_{\substack{x < b \\ x \vee a = b}} \mu(0,x) = 0.$$

Finally, by taking the summation to the other side, we are left with Weisner's Theorem (7), which we will restate for clarity:

$$\mu(0,b) = -\sum_{\substack{x < b \\ x \vee a = b}} \mu(0,x)$$

□

Now we have proven Weisner's Theorem, which is a critical step in proving Zaslavsky's Theorem. Notice we did not need the fact that the lattices here have a top element, showing that this theorem would work for semi-lattices with no top element. The proof uses the top element z in $\bar{L}$ as the top element of the entire semi-lattice, since anything above z is inconsequential and is not used calculating values using Weisner's Theorem.

3.3 Deletion and Contraction

The last ideas we will present before introducing Zaslavsky's Theorem will be those of deletion and contraction. Here are the definitions:

Definition 7 *Given an arrangement $\mathcal{A}$ containing a hyperplane H, $\mathcal{A} - \{H\}$ is the corresponding arrangement without hyperplane H (thus, H is deleted from the arrangement).*

In this next figure, on the left we have the lattice for the central arrangement presented in Figure 1 with the corresponding Möbius function values. On the right, the lattice for the central arrangement with H_2 deleted, and the corresponding Möbius function values. The reader should be able to easily visualize the arrangement without H_2 drawn in.

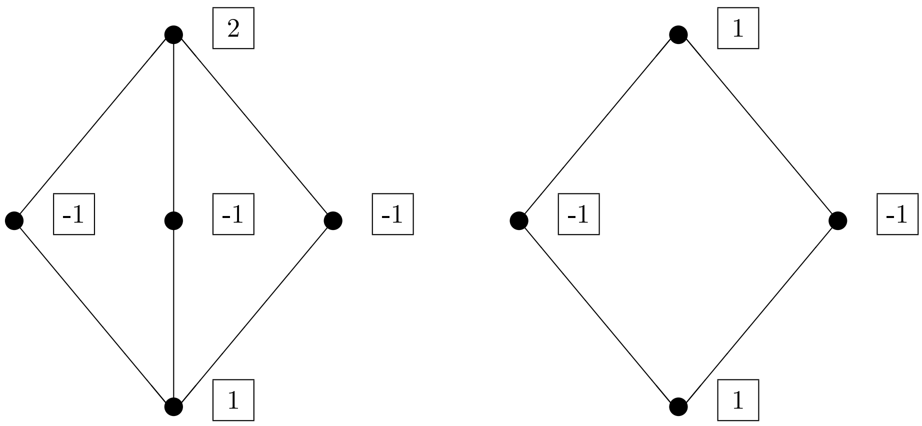
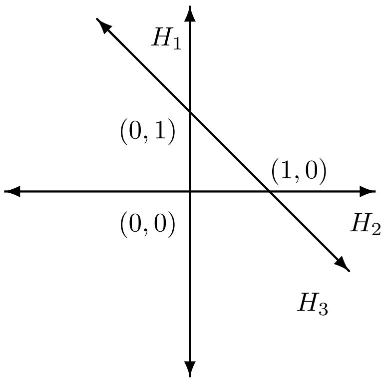


Figure 8

Definition 8 *Given an arrangement $\mathcal{A}$ containing a hyperplane H, $\mathcal{A}/\{H\}$ is the corresponding arrangement contracted to H. That is, H becomes the space, and hyperplanes intersecting it are noted in $\mathcal{A}/\{H\}$ and in the corresponding (semi-) lattice.*

Look again at Figure 2:



If we consider just the hyperplane labelled H_2, we find that H_1 intersects it at $(0,0)$, H_3 intersects it at $(1,0)$, and nothing else is notable. Thus, below is the lattice of $\mathcal{A}/\{H\}$, with points showing what they represent on the left, and the corresponding Möbius function values in the box on the right:

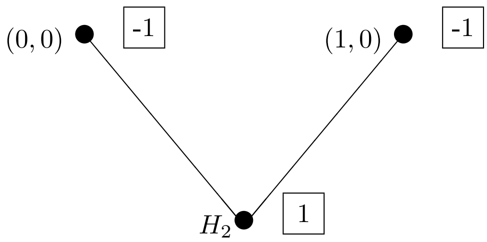


Figure 9

Now we have introduced all of the necessary material to state and prove Zaslavsky's Theorem.

4 Zaslavsky's Theorem

4.1 The Solution

Once again, our problem is given an arrangement $\mathcal{A}$ to be able to easily calculate how many regions the vector space is split into. The introduction set up the solution to this problem, the result proven by Zaslavsky. We set up a (semi-)lattice detailing the intersections of the hyperplanes in $\mathcal{A}$, and compute the Möbius function values of the nodes. Finally, here is an informal description of Zaslavsky's Theorem:

Zaslavsky's Theorem:
The number of regions into which a space is split by the arrangement $\mathcal{A}$ can be found by adding together the absolute values of the Möbius function values given to the corresponding (semi-)lattice.

We look at the two examples we have been dealing with to check this result.

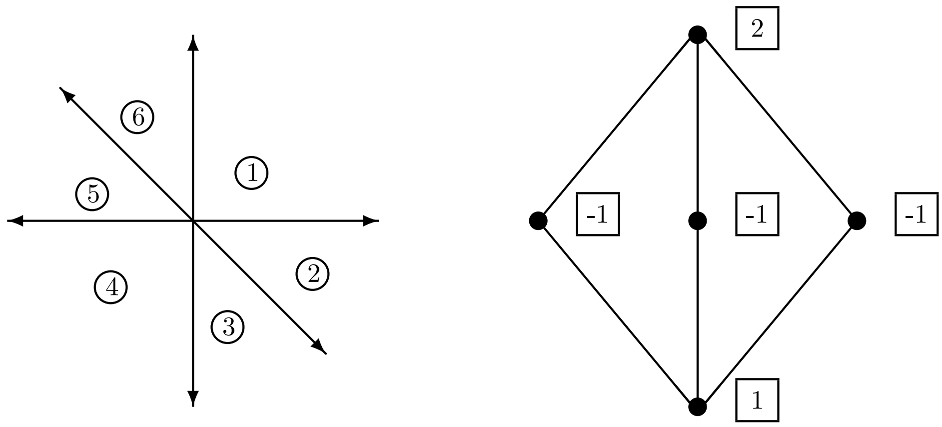


Figure 10

In Figure 10, the central arrangement on the left splits the plane into 6 regions (the circled numbers). The absolute values of the Möbius function values for the corresponding lattice on the right sum to 6.

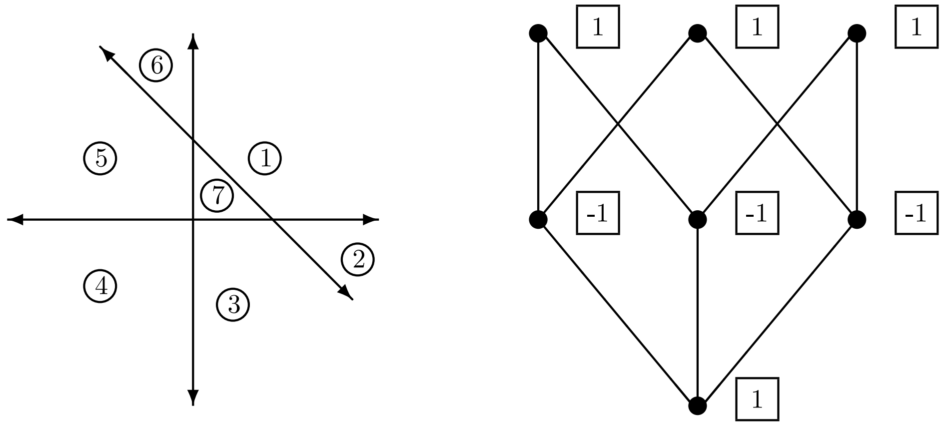


Figure 11

In Figure 11, The affine arrangement on the left splits the plane into 7 regions (the circled numbers). The absolute values of the Möbius function values for the corresponding lattice on the right sum to 7.

4.2 The Theorem

Theorem 3 (Zaslavsky's Theorem) *Given a vector space V and an arrangement $\mathcal{A}$ on V, the number of regions the vector space V is split into by $\mathcal{A}$ (denoted $N(\mathcal{A})$), can be expressed as follows:*

$$N(\mathcal{A}) = \sum_{x \in L(\mathcal{A})} (-1)^{r(x)} \mu(0, x). \tag{8}$$

Before we present the proof, let us verify that the statement above is the same as what we presented in words, that this indeed sums the absolute values of the Möbius function values of the (semi-)lattice for an arrangement. By inspection, the statement above does sum the Möbius function values up to a change in sign. So we can verify that this statement sums the absolute values of the Möbius function values by making sure we are always summing positive numbers.

Lemma 2 *Given a geometric (semi-)lattice $L(\mathcal{A})$, for every $x \in L(\mathcal{A})$,*

$$(-1)^{r(x)} \mu(0, x) > 0. \tag{9}$$

Proof: (by induction)

First, we check the first couple of cases.

Case 1: If x has rank 0, i.e. x is the bottom element of the lattice, then $\mu(x,x) = 1$ by the definition of the Möbius function, and $(-1)^0 = 1 > 0$.

Case 2: Any atom a_0 has rank 1, $\mu(0,a_0) = -1$, and $(-1)^1(\mu(0,a_0)) = 1 > 0$.

Now assume the lemma works for elements z with rank n. That is $(-1)^{r(z)}(\mu(0,z)) = (-1)^n(\mu(0,z)) > 0$.

We know $L(\mathcal{A})$ is a geometric (semi-)lattice, so every $x \in L$ is a join of atoms, and if x covers $x \wedge y$, then $x \vee y$ covers y.

Pick a (an atom of rank 1). For any $y \vee a = x$ where $x \neq y$, if x has rank n, then y has rank $n-1$ (or, equivalently, x is one dimension smaller than y). This is because since $L(\mathcal{A})$ is a geometric (semi-)lattice, a covers $a \wedge y$, which must be V, the bottom element ($x \neq y$, so $y \ngeq a$, and the only element lower than a is V), therefore $a \vee y = x$ covers y, so $r(x) = r(y) + 1$.

Now we use Weisner's Theorem: $a \in L(\mathcal{A})$, $a > 0$. Therefore, for all $x \geq a$,

$$\mu(0,x) = -\sum_{\substack{y < x \\ y \vee a = x}} \mu(0,y).$$

By our assumption, $(-1)^{r(y)}(\mu(0,y)) > 0$ for every y in the sum, since they all have rank n. So for every y in the sum, $\mu(0,y)$ has the same sign, so addition of all of them will also have the same sign. By Weisner's Theorem above, $\mu(0,x)$ has opposite sign, so if $(-1)^{r(y)}(\mu(0,y)) > 0$, we have $(-1)^{r(x)}(\mu(0,x)) = (-1)^{r(y)+1}(\mu(0,x)) > 0$.

So we have proven that $(-1)^{r(x)}(\mu(0,x)) > 0$ for the $(n+1)$ case, and our proof by induction is complete. $\square$

We have just verified that our algebraic representation of Zaslavsky's Theorem does indeed sum the absolute values of the Möbius function values for a geometric (semi-)lattice, and can now proceed with the proof of Zaslavsky's Theorem.

4.3 The Proof

Again, Zaslavsky's Theorem states that

$$N(\mathcal{A}) = \sum_{x \in L(\mathcal{A})} (-1)^{r(x)} \mu(0,x).$$

An important fact used for the proof of Zaslavsky's Theorem is the following identity:

$$N(\mathcal{A}) = N(\mathcal{A} - h) + N(\mathcal{A}/h). \tag{10}$$

The justification is best visualized in three dimensions, although it extends to any dimensional space:

Looking at an arrangement of hyperplanes involving the hyperplane h, some regions involve being cut by h, and some may not involve being cut by h. The regions that do not involve h stay unchanged in $\mathcal{A} - h$, and do not appear at all in $\mathcal{A}/h$. Some regions are cut into two by h, and each collapses into just one region in $\mathcal{A} - h$. Which regions are these? Take two regions next to each other in $\mathcal{A} - h$ that are each split into two regions in $\mathcal{A}$. There must be a hyperplane intersecting h in $\mathcal{A}$ that splits these regions up. Therefore, in $\mathcal{A}/h$ we get two regions, the region of h to the one side of the intersecting hyperplane, and the region of h to the other side of the intersecting hyperplane. Therefore, the two regions we lost going from $\mathcal{A}$ to $\mathcal{A} - h$ are counted in $\mathcal{A}/h$. The same argument can be made for any regions cut by h. Therefore, the number we lost going from $N(\mathcal{A})$ to $N(\mathcal{A} - h)$ is gained back in $N(\mathcal{A}/h)$, so that $N(\mathcal{A}) = N(\mathcal{A} - h) + N(\mathcal{A}/h)$.

We now proceed with the proof, which proceeds by induction on the number of hyperplanes in an arrangement.

Case 1: The arrangement $\mathcal{A}_1$ contains only one hyperplane, which splits the vector space into 2 regions. The lattice contains elements for the vector space, with Möbius function value 1, and the hyperplane, with Möbius function value -1, and $1 + \mid -1 \mid = 2$.

Now for an arrangement $\mathcal{A}$, assume the theorem works for all hyperplane arrangements with fewer hyperplanes than $\mathcal{A}$, and that $\mathcal{A}$ contains the hyperplane h. We know that $N(\mathcal{A}) = N(\mathcal{A} - h) + N(\mathcal{A}/h)$, and since by our assumption, Zaslavsky's Theorem works for $N(\mathcal{A} - h)$ and $N(\mathcal{A}/h)$ (since necessarily both $(\mathcal{A} - h)$ and $(\mathcal{A}/h)$ have less elements than $\mathcal{A}$), we can write the following:

$$N(\mathcal{A}) = \sum_{x \in L_{\mathcal{A}-h}} (-1)^{r(x)} \mu_{\mathcal{A}-h}(0, x) + \sum_{y \in L_{\mathcal{A}/h}} (-1)^{r_{\mathcal{A}/h}(y)} \mu_{\mathcal{A}/h}(\tilde{0}, y), \qquad (11)$$

where $\tilde{0}$ is the minimal element in $\mathcal{A}/h$ (therefore, h). Notice the rank function in $L(\mathcal{A})$ is the same as that in $L(\mathcal{A} - h)$. The ranks of the hyperplanes and their intersections do not change whether the arrangement contains h or not, although they certainly do in $\mathcal{A}/h$.

Our job now is to prove that (11) sums to Zaslavsky's formula. To do this, we split the elements in our lattice $L(\mathcal{A})$ into three categories:

1. Elements that are not bigger than or equal to h, so they do not interact at all with h. Therefore, they are all in $L(\mathcal{A} - h)$, and none of them are in $L(\mathcal{A}/h)$.

2. Elements that are bigger than or equal to h and are in $L(\mathcal{A} - h)$. So while they do interact with h, removing h does not cause these elements to disappear.

3. Elements that are bigger than or equal to h and not in $L(\mathcal{A}-h)$. These elements depend on the existence of h to exist, and are all in $L(\mathcal{A}/h)$.

We call these groups A (not to be confused with the hyperplane arrangement $\mathcal{A}$), B, and C, and define them algebraically like so:

- $A = \{x \in L \mid x \not\geq h\}$ $(\subseteq L(\mathcal{A}-h))$
- $B = \{y \in L \mid y \geq h,\ y \in L(\mathcal{A}-h)\}$
- $C = \{z \in L \mid z \geq h,\ z \notin L(\mathcal{A}-h)\}$

¿From these definitions, we see that in equation (11), the first summation adds the absolute values of the Möbius function values of the elements in A and in B, while the second summation adds the absolute values of the Möbius function values of the elements in B and in C. Therefore, elements in B are in both summations. Equation (11) could now be rewritten as:

$$\begin{aligned} N(\mathcal{A}) = \sum_{x\in A}(-1)^{r(x)}\mu_{\mathcal{A}-h}(0,x) + \sum_{y\in B}(-1)^{r(y)}\mu_{\mathcal{A}-h}(0,y) + \\ \sum_{y\in B}(-1)^{r_{\mathcal{A}/h}(y)}\mu_{\mathcal{A}/h}(\tilde{0},y) + \sum_{z\in C}(-1)^{r_{\mathcal{A}/h}(z)}\mu_{\mathcal{A}/h}(\tilde{0},z). \end{aligned} \tag{12}$$

If we can prove that the equation above equals the statement in Zaslavsky's Theorem, we are done. Obviously, from the way we defined A, B and C, each are disjoint, and their union equals the whole lattice $L(\mathcal{A})$. We need to prove that the equation above is equal to summing the original Möbius function values of all elements in $L(\mathcal{A})$ exactly once. We do this by proving three results:

1. $$\sum_{x\in A}(-1)^{r(x)}\mu_{\mathcal{A}-h}(0,x) = \sum_{x\in A}(-1)^{r(x)}\mu(0,x)$$

2. $$\sum_{z\in C}(-1)^{r_{\mathcal{A}/h}(z)}\mu_{\mathcal{A}/h}(\tilde{0},z) = \sum_{z\in C}(-1)^{r(z)}\mu(0,z)$$

3. $$\sum_{y\in B}(-1)^{r(y)}\mu_{\mathcal{A}-h}(0,y) + \sum_{y\in B}(-1)^{r_{\mathcal{A}/h}(y)}\mu_{\mathcal{A}/h}(\tilde{0},y) = \sum_{y\in B}(-1)^{r(y)}\mu(0,y)$$

Where $\mu(x, y)$ denotes the Möbius function on elements x, y in the original lattice $L(\mathcal{A})$. These claims are stated and proved in order of difficulty. Claim 1, by far the easiest to prove, is now restated and proved.

Claim 1 *In a (semi-)lattice $L(\mathcal{A})$ for a hyperplane arrangement $\mathcal{A}$, with A as defined before,*

$$\sum_{x \in A} (-1)^{r(x)} \mu_{\mathcal{A}-h}(0, x) = \sum_{x \in A} (-1)^{r(x)} \mu(0, x).$$

Proof: This result is trivial. Since elements in A do not depend on the hyperplane h at all (that is, the element is not a result of an intersection involving h), taking away h in the arrangement does not affect the rank of that element. Since these elements are not connected to h at all, their Möbius function values are not affected either. $\square$

Claim 2 *In a (semi-)lattice $L(\mathcal{A})$ for a hyperplane arrangement $\mathcal{A}$, with C as defined before,*

$$\sum_{z \in C} (-1)^{r_{\mathcal{A}/h}(z)} \mu_{\mathcal{A}/h}(\tilde{0}, z) = \sum_{z \in C} (-1)^{r(z)} \mu(0, z).$$

Proof: Assume we have an element $z \in C$, so z is neccesarily equal to a join of atoms, one of which is h.

Claim 2a: *There exists a unique element $w < z$ such that $h \vee w = z$.*
Proof: The element z is a join of atoms $\{h_1, h_2, \ldots h_n, h\}$. Consider this list of atoms without h (just $\{h_1, h_2, \ldots h_n\}$). Since the original join of atoms described some element z, the join of this list must describe some element $w < z$, such that $h \vee w = z$. We could not have $w = z$, since z is an element of C and therefore depends on h. Now, suppose there was some element u such that $u \vee h = z$. Since $L(\mathcal{A})$ is a geometric (semi-)lattice, $u = \bigvee \{\text{Atoms} \leq u\}$. Also, u must contain all atoms less than z except h, since $u \vee h = z$, therefore, $u = \{h_1, h_2, \ldots h_n\} = w$. So there is only one element $w < z$ such that $h \vee w = z$. $\square$

Now, we look once again at Weisner's Theorem: $h \in C$, $h > 0$. Therefore, $\forall z \geq h$,

$$\mu(0, z) = - \sum_{\substack{y < z \\ y \vee h = z}} \mu(0, y).$$

As we just showed, there is only one element w such that $w < z$, $w \vee h = z$, therefore Weisner's Theorem gives us this important fact:

$$-\mu(0, z) = \mu(0, w). \tag{13}$$

The next step is to prove that $\mu(0, w) = \mu(h, z)$, which will imply Claim 2.
Remark: It will be useful to prove that $\{\text{Atoms} \leq z\} = \{\text{Atoms} \leq w\} \cup \{h\}$.
Proof: We know that $L(\mathcal{A})$ is a geometric (semi-)lattice, therefore $a \in L(\mathcal{A}) \Rightarrow a = \bigvee \{\text{Atoms} \leq a\}$. Suppose h' is an atom, $h' \leq z$, $h' \neq h$, and $h' \not\leq w$. Then

$z \geq h' \vee w$, since $z > w$. But z covers w, therefore $z = h' \vee w$. This contradicts the fact that z is in C, since z must depend on h. Therefore, there are no atoms $\leq z$ that are $\nleq w$ besides h. □

Claim 2b: *Given the map $\phi : x \to x \vee h$, the map of the interval $[0, w] \to [h, z]$ is an isomorphism.*

Proof: The map $\phi : x \to x \vee h$ is onto. Consider an arbitrary q such that $h \leq q \leq z$. Since z is an element dependent on the hyperplane h, every element in $[h, z]$ must be dependent on h also, and therefore in C. This gives us that $q \in C$. Therefore, using an argument similar to Claim 2a, we can prove that $q = w' \vee h$ for some $w' \leq w$. This proves that the map is onto.

To prove that the map $\phi : x \to x \vee h$ is injective, we need to prove that for any $q \in C$, $q = w' \vee h = w'' \vee h \Rightarrow w' = w''$. Suppose $q = w' \vee h = w'' \vee h$ and $w' \neq w''$. Since L is a geometric (semi-)lattice, and since h covers $w' \wedge h$ (since $w' \wedge h$ must be the minimal element 0), w' must cover $q = w' \vee h$. Same with w''. So w' and w'' have rank one less than the rank of q, and q is above both of them. Therefore, q must be the least upper bound of w' and w'', which means that $q = w' \vee w'' \leq w$. But, since $q \in C$, this implies that $w \geq h$, which is a contradiction. So the map must be injective.

Finally, we need to prove the statement $x \vee h \leq y \vee h \Leftrightarrow x \leq y$. Proving the statement ($\Leftarrow$) is trivial, since we stated in the last section that this map is a closure operator.

To prove the statement ($\Rightarrow$), suppose $x \vee h \leq y \vee h$. Then, if $x \vee h$ is a join of the hyperplanes $\{h_1, h_2, \ldots h_n, h\}$, $y \vee h$ must be a join of all of these hyperplanes and maybe more. Therefore, x is a join of $\{h_1, h_2, \ldots h_n\}$, and y is a join of these and maybe more, so $x \leq y$.

By all these facts, we know that given the map $\phi : x \to x \vee h$, the map of the path $[0, w] \to [h, z]$ is an isomorphism. □

Given the fact that the paths are an isomorphism, we have this key fact:

$$\mu(0, w) = \mu(h, z). \tag{14}$$

Combining our results, we have:

$$-\mu(0, z) = \mu(0, w) = \mu(h, z). \tag{15}$$

We have now proven this fact for any $z \in C$, and we have already proved that

$$(-1)^{r(x)}\mu(0, x) > 0.$$

Since h is our minimal element ($\tilde{0}$) in C, we have our result:

$$\sum_{z \in C} (-1)^{r_{\mathcal{A}/h}(z)} \mu_{\mathcal{A}/h}(\tilde{0}, z) = \sum_{z \in C} (-1)^{r(z)} \mu(0, z).$$

□

Finally, by proving the third claim, we will have proved Zaslavsky's Theorem.

Claim 3 *In a (semi-)lattice $L(\mathcal{A})$ for a hyperplane arrangement $\mathcal{A}$, with B as defined before,*

$$\sum_{y\in B}(-1)^{r(y)}\mu_{\mathcal{A}-h}(0,y)+\sum_{y\in B}(-1)^{r_{\mathcal{A}/h}(y)}\mu_{\mathcal{A}/h}(\tilde{0},y)=\sum_{y\in B}(-1)^{r(y)}\mu(0,y).$$

Proof: Obviously, $\sum_{y\in B}(-1)^{r_{\mathcal{A}/h}(y)}\mu_{\mathcal{A}/h}(\tilde{0},y)=\sum_{y\in B}(-1)^{r(y)-1}\mu(h,y)$, so we can now write this sum as:

$$\sum_{y\in B}(-1)^{r(y)}\mu_{\mathcal{A}-h}(0,y)+\sum_{y\in B}(-1)^{r(y)-1}\mu(h,y),=\sum_{y\in B}(-1)^{r(y)}\mu(0,y).$$

To show that these sums are equal, it will suffice to show that for any element $z\in B$,

$$\mu_{\mathcal{A}-h}(0,z)-\mu(h,z)=\mu(0,z).$$

Or equivalently:

$$\mid\mu_{\mathcal{A}-h}(0,z)\mid+\mid\mu(h,z)\mid=\mid\mu(0,z)\mid,$$

since we know by Claim 2 that $\mu(h,z)$ must have sign opposite to the others.

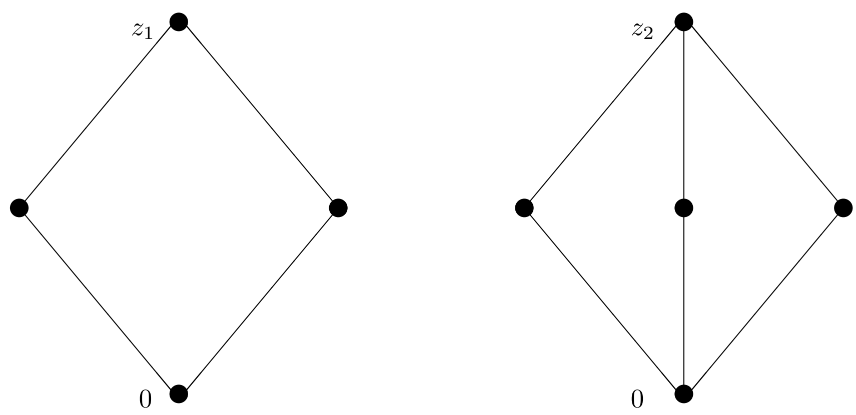
First, we will prove an important lemma:

Lemma 3 *For any lattice L, and any element $z\in L$,*

$$\mu(0,z)=\sum_{S}(-1)^{|S|},$$

where the sum is over all subsets S of atoms $\leq z$ such that $\bigvee S=z$.

Examples:



Figures 12 and 13

In Figure 12 on the left, there is only one S, the set containing the two atoms that have z_1 as their join. The reader can check that $\mu(0, z_1) = 1$, and that $(-1)^{|S|} = (-1)^2 = 1$, so the result checks.

In Figure 13 on the right, there are four subsets S that will work. If we take any 2 atoms, their join is z_2, and there are three such subsets of size 2. Also, there is the subset containing all of the atoms which is of size 3. The reader can check that $\mu(0, z_2) = 2$, and we have $(-1)^2 + (-1)^2 + (-1)^2 + (-1)^3 = 3 - 1 = 2$, so the result again holds.

Proof: This is a proof by induction.
Case 1: If z equals the minimal element 0, then $\mu(z, z) = 1$ by definition. There is only the empty subset that joins to 0, and $(-1)^0 = 1$. So this case works.

Now, assume the lemma is true for $y < z$. By definition of the Möbius function, we have

$$\mu(0, z) = -\sum_{\substack{y \\ 0 \le y < z}} \mu(0, y).$$

By our inductive assumption, this can be rewritten as:

$$\mu(0, z) = -\sum_{\substack{y \\ 0 \le y < z}} \sum_{\substack{S \\ \bigvee S = y}} (-1)^{|S|},$$

which simplifies to:

$$\mu(0, z) = -\sum_{\substack{S \\ \bigvee S < z}} (-1)^{|S|}.$$

One way to get all subsets S of atoms less than or equal to z, such that $\bigvee S < z$ is to take <u>all</u> subsets S of atoms less than or equal to z (so that $\bigvee S \le z$) and remove those whose join is equal to z. That is:

$$\mu(0, z) = -\Big(\sum_{\substack{all\ S \\ \bigvee S \le z}} (-1)^{|S|} - \sum_{\substack{S \\ \bigvee S = z}} (-1)^{|S|}\Big).$$

Now, since the atoms less than or equal to z form a finite set, the proof of the lemma now relies on the following sublemma:

Sublemma 1 *For any finite set X, and for all subsets S of X (including the empty set $\emptyset$), we have*

$$\sum_{all\ S \subseteq X} (-1)^{|S|} = 0.$$

Proof: Say X contains n elements. We have $\left(\begin{array}{c} n \\ 0 \end{array}\right)$ sets of size 0, $\left(\begin{array}{c} n \\ 1 \end{array}\right)$ sets of size 1, and so on until $\left(\begin{array}{c} n \\ n \end{array}\right)$. So we get the sum $\sum_{k=0}^{n}(-1)^k \left(\begin{array}{c} n \\ k \end{array}\right)$, which we know equals zero by the binomial theorem. □

Now, restating our last equation,

$$\mu(0,z) = -\Big(\sum_{\substack{all\ S \\ \bigvee S \leq z}} (-1)^{|S|} - \sum_{\substack{S \\ \bigvee S = z}} (-1)^{|S|} \Big).$$

We see that the first summation equals zero, so we get

$$\mu(0,z) = \sum_{\substack{S \\ \bigvee S = z}} (-1)^{|S|}.$$

□

Now to see why the lemma gives us the statement we need to prove the third claim:

Lemma 4 *For any lattice L, given an atom $h \in L$, and an element $z \in L$ such that $z \geq h$, we have*

$$\mu(0,z) = \mu_{\mathcal{A}-h}(0,z) - \mu(h,z).$$

Or, equivalently,

$$\mid \mu(0,z) \mid = \mid \mu_{\mathcal{A}-h}(0,z) \mid + \mid \mu(h,z) \mid .$$

Proof: By lemma 3, we have

$$\mu(0,z) = \sum_{\substack{S \\ \bigvee S = z}} (-1)^{|S|}.$$

We can split the different S's into two groups, one group where each S contains the hyperplane h, and the other where no S contains the hyperplane h.

$$\mu(0,z) = \sum_{\substack{h \notin S \\ \bigvee S = z}} (-1)^{|S|} + \sum_{\substack{h \in S \\ \bigvee S = z}} (-1)^{|S|}.$$

Since the first summand adds subsets S containing atoms less than or equal to z but not containing h, by lemma 3 we have:

$$\sum_{\substack{h \notin S \\ \bigvee S = z}} (-1)^{|S|} = \mu_{\mathcal{A}-h}(0,z).$$

So it suffices to show that the second summand equals $\mu(h,z)$. We now manipulate this summand in the following ways:

$$\sum_{\substack{h \in S \\ \bigvee S = z}} (-1)^{|S|} = -\sum_{\substack{h \notin S \\ (\bigvee S) \vee h = z}} (-1)^{|S|} = -\sum_{T \subseteq atoms\ of\ [h,z]} \sum_{S} (-1)^{|S|},$$

where in the last statement, in the second sum each S corresponds to T when its elements are joined with h. The fact that these sums are equal should be intuitively clear, and therefore the proof is not included.

Lemma 4 now relies on the following lemma.

Lemma 5 *For a lattice L and subsets S and T, where each S corresponds to T when its elements are joined with h,*

$$\sum_{S} (-1)^{|S|} = (-1)^{|T|}.$$

Proof: Pick an arbitrary T, where $T = \{t_1, t_2, \ldots, t_f\}$, a collection of atoms of $[h,z]$. Also, define A_i to be the set of atoms of L less than t_i minus the atom h ($A_i = \{atoms\ of\ L < t_i\} - \{h\}$). Now we can write

$$\sum_{S} (-1)^{|S|} = \sum_{S_1 \subseteq A_1, S_1 \neq \emptyset} (-1)^{|S_1|} \sum_{S_2 \subseteq A_2, S_2 \neq \emptyset} (-1)^{|S_2|} \cdots \sum_{S_f \subseteq A_f, S_f \neq \emptyset} (-1)^{|S_f|},$$

since when you join S with the hyperplane h you get exactly T.

Each A_i is a finite set, and from Sublemma 1 we have:

$$\sum_{all\ S_i \subseteq A_i} (-1)^{|S_i|} = 0$$

when we include the empty set. In each of the summands above, we only do not include the empty set, but include every other subset of each A_i. So therefore

$$\sum_{S_i \subseteq A_i, S_i \neq \emptyset} (-1)^{|S_i|} = -1$$

for each S_i, since the empty set contributes a positive 1 to the sum. So finally we have

$$\sum_{S} (-1)^{|S|} = \sum_{S_1 \subseteq A_1, S_1 \neq \emptyset} (-1)^{|S_1|} \sum_{S_2 \subseteq A_2, S_2 \neq \emptyset} (-1)^{|S_2|} \cdots \sum_{S_f \subseteq A_f, S_f \neq \emptyset} (-1)^{|S_f|} = (-1)^f = (-1)^{|T|},$$

which proves the lemma. $\square$

Using this lemma, we now get

$$-\sum_{T \subseteq atoms\ of\ [h,z]} \sum_{S} (-1)^{|S|} = -\sum_{T \subseteq atoms\ of\ [h,z]} (-1)^{|T|} = -\mu(h,z),$$

where the last step uses Lemma 3.

Therefore, we have now proven Lemma 4, that $\mu_{\mathcal{A}-h}(0,z)-\mu(h,z)=\mu(0,z)$, which proves the last claim needed to prove Zaslavsky's Theorem, so we are done. $\Box$

To restate for clarity, Zaslavsky's Theorem will work in the same way for arrangements in higher dimensions. Simply figure out the set of intersections for the arrangement, construct the (semi-)lattice, write the Möbius function values for each node, and add the absolute values. Arrangements with intersections in more dimensions will simply have (semi-)lattices with more levels.

5 Applications of Zaslavsky's Theorem

In Orlik and Terao's book "Arrangements of Hyperplanes" ([2]), the authors start off by saying that the "humble origins of our subject" can be summed up by this problem:

> Show that n cuts can divide a cheese into as many as $(n+1)(n^2-n+6)/6$ pieces.
>
> Problem E 554. *Amer. Math. Monthly* **50** (1943), p. 59. *Proposed by J. L. Woodbridge, Philadelphia*

Using Zaslavsky's Theorem, we can easily prove a powerful result that will solve this problem.

Theorem 4 *For a vector space of dimension m, n hyperplanes (each of dimension $m-1$) can split the vector space into at most*

$$\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{m}$$

regions.

Proof: As Orlik and Terao state, to maximize the number of regions the hyperplanes split a vector space into, the hyperplanes must be in "general position." This means that any two hyperplanes of dimension $m-1$ intersect in a space of dimension $m-2$ that is distinct from all other intersections, that any three hyperplanes of dimension $m-1$ intersect in a space of dimension $m-3$ that is distinct from all other intersections, and so on.

For example, working in $\Re^3$, any two planes have a common line, and the lines are distinct, and and any two lines meet at a common point that is distinct. It is not hard to convince yourself that hyperplanes in general position indeed give the maximum number of regions.

For example, in Figure 1 we had three lines meeting at a common point, which split $\Re^2$ into six regions. Meanwhile, Figure 2 involved hyperplanes in general position. Any two lines meet at a common and distinct point, and this arrangement split $\Re^2$ into seven regions. In general, we can see that if three or more lines meet at a common point, moving hyperplanes until they are in general position will result in more regions, and it is obvious that this result generalizes to higher dimensions. We would not want any more than two hyperplanes intersecting in any area on our graph, as that would reduce the number of total regions in any dimension.

So in terms of the semi-lattice, we get one 0 element, then n atoms. Then, since any two hyperplanes of dimension $m-1$ meet in a common and distinct space of dimension $m-2$, we get $\binom{n}{2}$ elements of rank 2 in our semi-lattice. Since any three hyperplanes of dimension $m-1$ meet in a common and distinct space of dimension $m-3$, we get $\binom{n}{3}$ elements of rank 3 in our semi-lattice. And so on, until the end.

Because the hyperplanes are in general position, for an element $z \in L(\mathcal{A})$, there is only one subset S containing atoms less than or equal to z, such that $\bigvee S = z$. Therefore, by Lemma 3, we have $\mu(0, z) = \sum_S (-1)^{|S|} = \pm 1$.

Using Zaslavsky's Theorem, we add up the absolute values of each of the Möbius function values of the elements, which in this case is always 1, so we have

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{m}$$

regions. $\square$

The reader can check that $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3}$ equals $(n+1)(n^2 - n + 6)/6$, stated at the beginning of this section.

One last application, which will be stated but not proven, concerns determining how many bounded regions an arrangement splits a vector space into. For example, the central arrangement we have been looking at splits $\Re^2$ into no bounded regions, while the affine arrangement splits it into 1.

Theorem 5 *Given a vector space V and an arrangement $\mathcal{A}$ on V, the number of bounded regions the vector space V is split into by $\mathcal{A}$, can be expressed by:*

$$\left| \sum_{x \in L(\mathcal{A})} \mu(0, x) \right|. \tag{16}$$

That is, just adding up the Möbius function values without taking the absolute values, then taking the absolute value of that quantity gives the number of bounded regions. The reader can check this result for the examples we have been using. The proof can be found in [4].

We now end this paper with a more complicated example of the above theorem. Observe the following hyperplane arrangement in $\Re^2$:

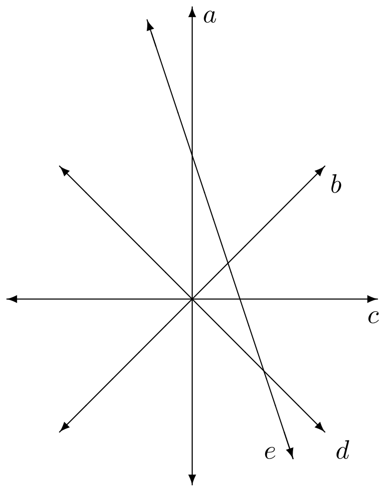
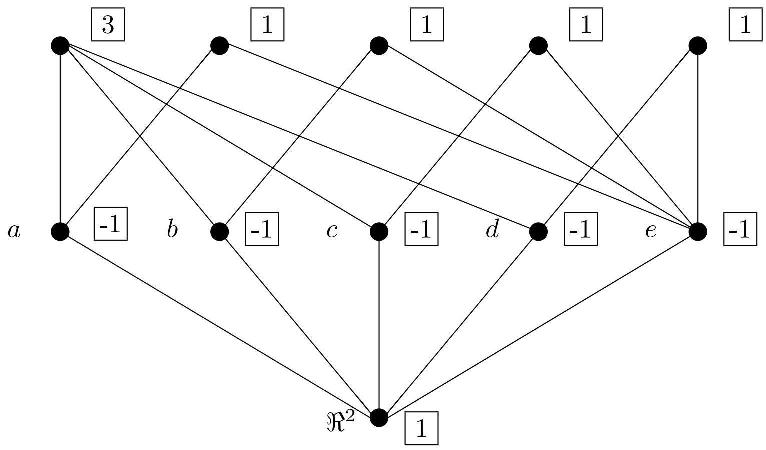


Figure 14

The arrangement has three bounded regions. Now, here is the lattice, with the atoms labelled according to which hyperplane they represent, and the Möbius function values placed to the right of the nodes in boxes.



Adding up these Möbius function values, we get the number 3, exactly the number of bounded regions of the arrangement presented in Figure 14.

References

1. Curtis Greene, "The Möbius Function of a Partially Ordered Set," Haverford College.

2. P. Orlik and H. Terao, "Arrangements of Hyperplanes," *Grundlehren der mathematischen Wissenschaften* **300**, Springer-Verlag, Berlin/New York, 1992.

3. http://hissa.nist.gov/dads/HTML/partialorder.html

4. T. Zaslavsky, "Facing up to Arrangements: Face-Count Formulas for Partitions of Space by Hyperplanes," *AMS Memoirs, v.1, no. 154*, 1975.

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