A topological proof of Eliaz's unified theorem of social choice theory

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Abstract

Recently Eliaz (2004) has presented a unified framework to study (Arrovian) social welfare functions and non-binary social choice functions based on the concept of *preference reversal*. He showed that social choice rules which satisfy the property of preference reversal and a variant of the Pareto principle are dictatorial. This result includes the Arrow impossibility theorem (Arrow (1963)) and the Gibbard-Satterthwaite theorem (Gibbard (1973), Satterthwaite (1975)) as its special cases. We present a concise proof of his theorem using elementary concepts of algebraic topology such as homomorphisms of homology groups of simplicial complexes induced by simplicial mappings.

Keywords: preference reversal; homomorphism; homology group; simplicial complex; simplicial mapping

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1 Introduction

Recently Eliaz (2004) has presented a unified framework to study (Arrovian) social welfare functions and non-binary social choice functions based on the concept of *preference reversal*. The preference reversal property is a condition (according to the expression in Eliaz (2004)) that if social relation (given by a social choice function or a social preference) between any two alternatives has been reversed, then someone must have exhibited the same reversal in his preference. He showed that social choice rules which satisfy the property of preference reversal and a variant of the Pareto principle are dictatorial. This result includes the Arrow impossibility theorem (Arrow (1963)) and the Gibbard-Satterthwaite theorem (Gibbard (1973), Satterthwaite (1975)) as its special cases. We present a concise proof of his theorem using elementary concepts of algebraic topology such as homomorphisms of homology groups of simplicial complexes induced by simplicial mappings.

Topological approaches to social choice problems have been initiated by Chichilnisky (1980). Her main result is an impossibility theorem that there exists no *continuous* social choice rule which satisfies *unanimity* and *anonymity.* This approach has been further developed by Chichilnisky (1979), (1982), Candeal and Indurain (1994), Koshevoy (1997), Lauwers (2004), Weinberger (2004), and so on. On the other hand, Baryshnikov (1993) and (1997) have presented a topological approach to the Arrow impossibility theorem (or general possibility theorem) in a discrete framework of social choice[1]. Our research is in line with the studies of topological approaches to discrete social choice problems initiated by him. In the next section we present expressions of binary social choice rules by simplicial complexes and simplicial mappings. In Section 3 we will prove the main results of this paper.

2 The model

There are m alternatives of a social problem, x_1, x_2, $\cdots$, x_m ($m \geqq 3$), and n individuals ($n \geqq 2$). The set of alternatives is denoted by A. m and n are finite integers. Individual preferences over these alternatives are complete, transitive and asymmetric. Individual i's preference is denoted by P_i. $x_i P_i x_j$ means that he prefers x_i to x_j.

A social choice rule which we will consider according to Eliaz (2004) is a rule that determines a social binary relation about each pair of alternatives corresponding to a combination of individual preferences. It may not be complete. We call such a social choice rule a *binary social choice rule.* It is abbreviated as BCR. We assume the universal (or unrestricted) domain condition for social binary choice rules[2]. We call a combination of individual preferences a *profile*. The profiles are denoted by $\mathbf{p}$, $\mathbf{p}'$ and so on. Individual i's preference at $\mathbf{p}'$ is denoted by P_i', and so on. A social binary relation generated by a BCR is denoted by R. We call it also a BCR. Let x_i and x_j be two distinct alternatives. $x_i R x_j$ means that x_i *relates to* x_j *according to BCR* R. On the other hand $x_i^{\neg} R x_j$ means that x_i *does not relate to* x_j *according to BCR* R. A BCR at a profile $\mathbf{p}$ is denoted by R, a BCR at $\mathbf{p}'$ is denoted by R', and so on.

Any BCR R is required to satisfy the following conditions.

Existence of a best alternative (BA) There exists an alternative $x_i \in A$ such that $x_i R x_j$ for all $x_j \in A \setminus \{x_i\}$. There may be multiple best alternatives.

[1]About surveys and basic results of topological social choice theories, see Mehta (1997) and Lauwers (2000).

[2]The universal domain condition means that the domain of individuals preferences for social binary choice rules is never restricted.

Acyclicality (AC) For every three alternatives x_i, x_j and x_k in A if x_iRx_j and $x_k^{\neg}Rx_j$, then $x_k^{\neg}Rx_i$.

Pareto efficiency (PAR) For every two alternatives x_i and x_j in A if all individuals prefer x_i to x_j, then either "x_iRx_j and $x_j^{\neg}Rx_i$", or "x_i and x_j are not related according to R ($x_i^{\neg}Rx_j$ and $x_j^{\neg}Rx_i$)".

Preference reversal (PR) For every two alternatives $x_i$and x_j in A if x_iRx_j, $x_j^{\neg}Rx_i$ but $x_jR'x_i$, then there exists (at least) one individual i such that $x_iP_ix_j$ and $x_jP_i'x_i$.

Dictator is defined as follows.

Dictator If, there exists an individual i such that for every pair of alternatives x_i and x_j the social relation is $x_j^{\neg}Rx_i$ whenever he prefers x_i to x_j, then he is the dictator of R.

As proved in Observation 1 of Eliaz (2004) AC is equivalent to the following Transitivity.

Transitivity (T) For every three alternatives x_i, x_j and x_k in A if x_iRx_j and x_jRx_k, then x_iRx_k.

Proof. (1) AC$\longrightarrow$ T: Assume that x_iRx_j, x_jRx_k but $x_i^{\neg}Rx_k$. Then, from x_jRx_k and $x_i^{\neg}Rx_k$ AC implies $x_i^{\neg}Rx_j$. It is a contradiction.

(2) T$\longrightarrow$ AC: Assume that x_iRx_j, $x_k^{\neg}Rx_j$ but x_kRx_i. Then, from x_kRx_i and x_iRx_j T implies x_kRx_j. It is a contradiction.

□

As noted by Eliaz (2004) if a BCR satisfies BA, AC and the Completeness (Condition C) (x_iRx_j or x_jRx_i), then it is an Arrovian social welfare function. In this interpretation AC means the transitivity of strict social preferences[3]. Eliaz (2004) showed that if a social welfare function satisfies BA, AC, PAR, C and Arrow's condition of *independence of irrelevant alternatives*, then it satisfies PR. If a BCR satisfies C, $x_j^{\neg}Rx_i$ is equivalent to x_iRx_j. Thus, the dictator in the above definition is the dictator for an Arrovian social welfare function.

On the other hand, if the unique alternative x_i satisfies x_iRx_j for all $x_j \in A \setminus \{x_i\}$ and all alternatives other than x_i are not mutually related according to a BCR R, then it is a social choice function which is a social choice rule that chooses one alternative corresponding to each profile. To be precise a social choice function chooses one alternative corresponding to a profile of *reported preferences* of individuals. If a social choice function does not give any incentive to every individual to report a preference which is different from his true preference, then it is *strategy-proof.* It was shown by Eliaz (2004) that a strategy-proof social choice function satisfies PR. If there exists the unique best alternative x_i

[3]From Lemma 1 of Baryshnikov (1993) we know that if individual preferences are strict orders, then the social preference is also a strict order under the transitivity, the Pareto principle and the independence of irrelevant alternatives.

for a BCR, then $x_j^{\neg}Rx_i$ means that x_j is not chosen by the social choice function derived from this BCR, and the dictator in the above definition is the dictator for the social choice function. Eliaz (2004) showed the theorem that if a BCR satisfies BA, AC, PAR and PR, it has the dictator. Then, the Arrow impossibility theorem that there exists the dictator for any social welfare function which satisfies BA, AC, C, PAR and the independence of irrelevant alternatives under the universal domain condition, and the Gibbard-Satterthwaite theorem that there exists the dictator for any social choice function which is onto (surjection) and strategy-proof under the universal domain condition are the special cases of his theorem.

PAR with BA implies the following condition[4].

Strong Pareto efficiency (SPAR) For every alternative x_i if all individuals prefer x_i to all other alternatives, then we have x_iRx_j and $x_j^{\neg}Rx_i$ for all $x_j \in A \setminus \{x_i\}$.

Now we consider topological expressions of individual preferences. We draw a circumference which represents the set of individual preferences by connecting $m!$ vertices $v_1, v_2, \cdots, v_{m!}$ by arcs[5]. For example, in the case of four alternatives, these vertices mean the following preferences.

$$v_1 : (1234),\ v_2 : (1243)\ v_3 : (1423),\ v_4 : (1432),\ v_5 : (1342),\ v_6 : (1324)$$

$$v_7 : (2134),\ v_8 : (2143)\ v_9 : (2413),\ v_{10} : (2431),\ v_{11} : (2341),\ v_{12} : (2314)$$

$$v_{13} : (3124),\ v_{14} : (3142)\ v_{15} : (3412)\ v_{16} : (3421),\ v_{17} : (3241),\ v_{18} : (3214)$$

$$v_{19} : (4123),\ v_{20} : (4132)\ v_{21} : (4312)\ v_{22} : (4321),\ v_{23} : (4231),\ v_{24} : (4213)$$

We denote a preference such that an individual prefers x_1 to x_2 to x_3 to x_4 by (1234), and so on. Notations for the cases with different number of alternatives are similar. Generally $v_1 \sim v_{(m-1)!}$ represent preferences such that the most preferred alternative for an individual is x_1, $v_{(m-1)!+1} \sim v_{2(m-1)!}$ represent preferences such that the most preferred alternative for an individual is x_2, and so on. In particular v_1 denotes a preference such that an individual prefers x_1 to x_2 to x_3 to $\cdots$ to x_m. It is denoted by $(123\cdots m)$.

Denote this circumference by S_i^1. S_i^1 in the case of three alternatives is depicted in Figure 1. The set of profiles of the preferences of n individuals is represented by the product space $S_i^1 \times \cdots \times S_i^1$ (n times). It is denoted by $(S_i^1)^n$. The 1-dimensional homology group of S_i^1 is isomorphic to the group of integers $\mathbb{Z}$, that is, $H_1(S_i^1) \cong \mathbb{Z}$. And the 1-dimensional homology group of $(S_i^1)^n$ is isomorphic to the direct product of n groups of integers $\mathbb{Z}^n$, that is, we have $H_1((S_i^1)^n) \cong \mathbb{Z}^n$. It is proved, for example, using the Mayer-Vietoris exact sequences[6].

[4] This term SPAR is not defined in Eliaz (2004).

[5] $m!$ denotes factorial of m.

$$m! = \Pi_{j=1}^m j = m(m-1)(m-2) \times \cdots \times 2 \times 1$$

[6] About homology groups and the Mayer-Vietoris exact sequences we referred to Tamura (1970) and Komiya (2001).

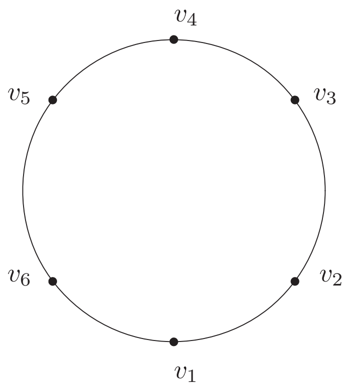

Figure 1: S_i^1

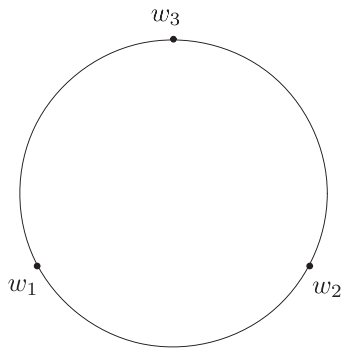

Figure 2: S^1

The social binary relation generated by a BCR is also represented by a circumference depicted in Figure 2. This circumference is drawn by connecting three vertices, w_1, w_2 and w_3 by arcs. These vertices mean the following social binary relations.

(1) w_2: binary relations such that $x_2 R x_j$ and $x_j^{\neg} R x_2$ for all $x_j \in A \setminus \{x_2\}$.

(2) w_3: binary relations such that $x_3 R x_j$ and $x_j^{\neg} R x_3$ for all $x_j \in A \setminus \{x_3\}$.

(3) w_1: all other social binary relations.

We call this circumference S^1. The 1-dimensional homology group of S^1 is also isomorphic to $\mathbb{Z}$, that is, $H_1(S^1) \cong \mathbb{Z}$.

Binary social choice rules are simplicial mappings. Binary social choice rules are denoted by $f: (S_i^1)^n \longrightarrow S^1$. Two adjacent vertices of S_i^1 span a 1-dimensional simplex. And any pair of two vertices of S^1 spans a 1-dimensional simplex. Thus, f is a simplicial mapping, and we can define the homomorphism of homology groups induced by f.

We define an inclusion mapping from S_i^1 to $(S_i^1)^n$ by $\Delta: S_i^1 \longrightarrow (S_i^1)^n$ under the assumption that all individuals have the same preferences, and define an inclusion mapping when the profile of preferences of individuals other than one individual (denoted by i) is fixed at some profile by $i_i: S_i^1 \longrightarrow (S_i^1)^n$. The homomorphisms of homology groups induced by these inclusion mappings are as follows.

$$\Delta_*: \mathbb{Z} \longrightarrow \mathbb{Z}^n: h \longrightarrow (h, h, \cdots, h),\ h \in \mathbb{Z}$$

$$i_{i*}: \mathbb{Z} \longrightarrow \mathbb{Z}^n: h \longrightarrow (0, \cdots, 0, h, 0, \cdots, 0),\ h \in \mathbb{Z} \text{ (only the } i\text{-th component is } h)$$

From these definitions we obtain the following relation about Δ_* and i_{i*} at any profile.

$$\Delta_* = \sum_{i=1}^{n} i_{i*} \tag{1}$$

Let us denote the homomorphism of homology groups induced by f by $f_* : (\mathbb{Z})^n \longrightarrow \mathbb{Z}$.

Binary social choice rules for different profiles are homotopic. f for a fixed profile of preferences of individuals other than i (denoted by $f|_{\mathbf{p}_{-i}}$) and f for another fixed profile of their preferences (denoted by $f|_{\mathbf{p}'_{-i}}$) are homotopic. Thus, the homomorphisms of homology groups induced by them are isomorphic. Denote two profiles of individuals other than i by $\mathbf{p}_{-i}$ and $\mathbf{p}'_{-i}$. Then, the homotopy between $f|_{\mathbf{p}_{-i}}$ and $f|_{\mathbf{p}'_{-i}}$ is

$$f_t = \frac{tf|_{\mathbf{p}_{-i}} + (1-t)f|_{\mathbf{p}'_{-i}}}{|tf|_{\mathbf{p}_{-i}} + (1-t)f|_{\mathbf{p}'_{-i}}|}\ (0 \leqq t \leqq 1)$$

It is well defined since $f|_{\mathbf{p}_{-i}}$ and $f|_{\mathbf{p}'_{-i}}$ are not anti-podal.

The composite function of i_i and f is denoted by $f \circ i_i :\ S_i^1 \longrightarrow S^1$, and its induced homomorphism of homology groups satisfies $(f \circ i_i)_* = f_* \circ i_{i*}$, for all i. The composite function of Δ and f is denoted by $f \circ \Delta :\ S_i^1 \longrightarrow S^1$, and its induced homomorphism of homology groups satisfies $(f \circ \Delta)_* = f_* \circ \Delta_*$. From (1) we obtain

$$(f \circ \Delta)_* = \sum_{i=1}^{n} (f \circ i_i)_* \tag{2}$$

3 The main results

In this section we will prove the following theorem by Eliaz (2004).

Theorem 1. *There exists the dictator for any BCR which satisfies BA, AC, PAR and PR.*

First we show the following lemma which will be used below.

Lemma 1. *Suppose that a BCR satisfies BA, AC, PAR and PR, and has no dictator. When the preference of one individual (denoted by i) is* $(234\cdots m1)$, *and the preferences of all other individuals are* v_1, *then we have*

$x_1 R x_j$ *and* $x_j^{\neg} R x_1$ *for all* $x_j \in A \setminus \{x_1, x_2\}$

Proof. **Step 1:**

Note that v_1 represents a preference $(123\cdots m)$. By PAR we have

$$x_2Rx_j \text{ (or } x_2^{\neg}Rx_j\text{) and } x_j^{\neg}Rx_2 \text{ for all } x_j \in A \setminus \{x_1, x_2\} \tag{3}$$

By BA there are the following three cases about x_1 and x_2[7].

(1) Case 1: x_2Rx_1 and $x_1^{\neg}Rx_2$.

(2) Case 2: x_1Rx_2 and $x_2^{\neg}Rx_1$.

(3) Case 3: x_1Rx_2 and x_2Rx_1.

It will be proved that in Case 1 individual i is the dictator. In Step 1 we consider this case. By PR we have $x_1^{\neg}Rx_2$ so long as individual i prefers x_2 to x_1. Then, we say that individual i is *decisive* for x_2 against x_1. Let x_j and x_k ($x_k \neq x_j$) be alternatives other than x_1 and x_2, and consider the following profile.

(1) Individual i prefers x_k to x_2 to x_1 to x_j to all other alternatives.

(2) Other individuals prefer x_1 to x_j to x_k to x_2 to all other alternatives.

By PR we have $x_1^{\neg}Rx_2$. And by PAR we have

(1) x_1Rx_j (or $x_1^{\neg}Rx_j$) and $x_j^{\neg}Rx_1$, and x_1Rx_l (or $x_1^{\neg}Rx_l$) and $x_l^{\neg}Rx_1$ for all $x_l \in A \setminus \{x_1, x_2, x_j, x_k\}$.

(2) x_kRx_2 (or $x_k^{\neg}Rx_2$) and $x_2^{\neg}Rx_k$, and x_kRx_l (or $x_k^{\neg}Rx_l$) and $x_l^{\neg}Rx_k$ for all $x_l \in A \setminus \{x_1, x_2, x_j, x_k\}$.

BA and AC imply that we have x_kRx_l and $x_l^{\neg}Rx_k$ for all $x_l \in A \setminus \{x_k\}$[8]. Then, by PR we have $x_j^{\neg}Rx_k$ so long as individual i prefers x_k to x_j, and so individual i is decisive for x_k against x_j. Note that x_j and x_k are arbitrary. Next consider the following profile.

(1) Individual i prefers x_2 to x_k to x_j to all other alternatives.

(2) Other individuals prefer x_j to x_2 to x_k to all other alternatives.

By PR we have $x_j^{\neg}Rx_k$. And by PAR we have

x_2Rx_k (or $x_2^{\neg}Rx_k$) and $x_k^{\neg}Rx_2$, and x_2Rx_l (or $x_2^{\neg}Rx_l$) and $x_l^{\neg}Rx_2$ for all $x_l \in A \setminus \{x_2, x_j, x_k\}$.

BA and AC imply that we have x_2Rx_l and $x_l^{\neg}Rx_2$ for all $x_l \in A \setminus \{x_2\}$. Then, by PR we have $x_j^{\neg}Rx_2$ so long as individual i prefers x_2 to x_j, and so individual i is decisive for x_2 against x_j. Next consider the following profile.

[7] If $x_1^{\neg}Rx_2$ and $x_2^{\neg}Rx_1$, then there exists no best alternative.

[8] BA implies x_kRx_l for all $x_l \in A \setminus \{x_k\}$, and from AC with $x_1^{\neg}Rx_2$, $x_j^{\neg}Rx_1$, $x_2^{\neg}Rx_k$ and $x_l^{\neg}Rx_k$ ($x_l \in A \setminus \{x_1, x_2, x_j, x_k\}$) we have $x_l^{\neg}Rx_k$ for all $x_l \in A \setminus \{x_k\}$.

(1) Individual i prefers x_k to x_j to x_2 to all other alternatives.

(2) Other individuals prefer x_j to x_2 to x_k to all other alternatives.

By PR we have $x_j^{\neg} R x_k$. And by PAR we have

$$x_j R x_2 \text{ (or } x_j^{\neg} R x_2\text{) and } x_2^{\neg} R x_j, \text{ and } x_j R x_l \text{ (or } x_j^{\neg} R x_l\text{) and } x_l^{\neg} R x_j \text{ for all } x_l \in A \setminus \{x_2, x_j, x_k\}.$$

BA and AC imply that we have $x_k R x_l$ and $x_l^{\neg} R x_k$ for all $x_l \in A \setminus \{x_k\}$. Then, by PR we have $x_2^{\neg} R x_k$ so long as individual i prefers x_k to x_2, and so individual i is decisive for x_k against x_2. By similar procedures we can show that individual i is decisive for x_1 against x_j, and is decisive for x_k against x_1. Finally consider the following profile.

(1) Individual i prefers x_1 to x_k to x_2 to all other alternatives.

(2) Other individuals prefer x_2 to x_1 to x_k to all other alternatives.

By PR we have $x_2^{\neg} R x_k$. And by PAR we have

$$x_1 R x_k \text{ (or } x_1^{\neg} R x_k\text{) and } x_k^{\neg} R x_1, \text{ and } x_1 R x_l \text{ (or } x_1^{\neg} R x_l\text{) and } x_l^{\neg} R x_1 \text{ for all } x_l \in A \setminus \{x_1, x_2, x_k\}.$$

BA and AC imply that we have $x_1 R x_l$ and $x_l^{\neg} R x_1$ for all $x_l \in A \setminus \{x_1\}$. Then, by PR we have $x_2^{\neg} R x_1$ so long as individual i prefers x_1 to x_2, and individual i is decisive for x_1 against x_2. Therefore, individual i is the dictator[9].

Step 2:

Next let us consider Case 2 and 3. From (3) we have $x_j^{\neg} R x_2$ for all $x_j \in A \setminus \{x_1, x_2\}$. Then in both Case 2 and 3, $x_1 R x_2$ and AC imply

$$x_j^{\neg} R x_1 \text{ for all } x_j \in A \setminus \{x_1, x_2\}$$

By BA in Case 2 we obtain

$$x_1 R x_j \text{ and } x_j^{\neg} R x_1 \text{ for all } x_j \in A \setminus \{x_1\}.$$

And in Case 3 we have[10]

$$x_1 R x_2,\ x_2 R x_1,\ x_1 R x_j \text{ and } x_j^{\neg} R x_1 \text{ for all } x_j \in A \setminus \{x_1, x_2\}. \tag{4}$$

Therefore, we get the conclusion of this lemma. □

[9]We can show that individual i is the dictator in Case 1 when there are only three alternatives by similar procedures.

[10]By BA we obtain

$$x_1 R x_j \text{ for all } x_j \in A \setminus \{x_1\}, \text{ or } x_2 R x_j \text{ for all } x_j \in A \setminus \{x_2\}$$

Then, AC or T(Transitivity) implies (4).

By SPAR we obtain the correspondences from the vertices of S_i^1 to the vertices of S^1 by $f \circ \Delta$ as follows.

$$v_1 \sim v_{(m-1)!} \longrightarrow w_1,\ v_{(m-1)!+1} \sim v_{2(m-1)!} \longrightarrow w_2,\ v_{2(m-1)!+1} \sim v_{3(m-1)!} \longrightarrow w_3$$

All other vertices correspond to w_1. Sets of 1-dimensional simplices included in S_i^1 which are 1-dimensional cycles are only the following z and its counterpart $-z$.

$$z = < v_1, v_2 > + < v_2, v_3 > + \cdots + < v_{m!-1}, v_{m!} > + < v_{m!}, v_1 >$$

Since S_i^1 does not have a 2-dimensional simplex, z is a representative element of homology classes of S_i^1. z is transferred by $(f \circ \Delta)_*$ to the following z'.

$$z' =< w_1, w_2 > + < w_2, w_3 > + < w_3, w_1 >$$

This is a cycle of S^1. Therefore, we have

$$(f \circ \Delta)_* \neq 0 \tag{5}$$

Now we show the following lemma.

Lemma 2. *If a BCR satisfies BA, AC, PAR and PR, and has no dictator, then we obtain*

$$(f \circ i_i)_* = 0 \quad \textit{for all } i \tag{6}$$

Proof. By SPAR when the preference of every individual other than one individual (denoted by i) is fixed at v_1, the correspondences from the preference of individual i to the social binary relation from v_1 to $v_{(m-1)!}$ are as follows.

$$v_1 \sim v_{(m-1)!} \longrightarrow w_1$$

Lemma 1 implies that the correspondence from $(234 \cdots m1)$ to the social binary relation is as follows.

$$(234 \cdots m1) \longrightarrow w_1, \text{ and we have } x_1 R x_j,\ x_j \neg R x_1 \text{ for all } x_j \in A \setminus \{x_1, x_2\}$$

Then, PR implies that x_3 is never the unique best alternative for BCR so long as the most preferred alternative for all individuals other than i is x_1 regardless of the preference of individual i, and so the preference of individual i corresponds to w_1 or w_2. Thus, we obtain the following correspondences.

$$v_{(m-1)!+1} \sim v_{m!} \longrightarrow w_1 \text{ or } w_2$$

Sets of 1-dimensional simplices included in S_i^1 which are 1-dimensional cycles are only the following z and its counterpart $-z$.

$$z = < v_1, v_2 > + < v_2, v_3 > + \cdots + < v_{m!-1}, v_{m!} > + < v_{m!}, v_1 >$$

Since S_i^1 does not have a 2-dimensional simplex, z is a representative element of homology classes of S_i^1. z is transferred by $(f \circ i_l)_*$ to the following z'.

$$z' =< w_1, w_2 > + < w_2, w_1 >= 0 \text{ or } z' =< w_1, w_1 >= 0$$

Therefore, we have $(f \circ i_i)_* = 0$ for all i. □

The conclusion of this lemma contradicts (2) and (5). Therefore, we have shown Theorem 1. We call the property expressed in (6) the *non-surjectivity of individual inclusion mappings*. Then, Theorem 1 is a special case of the following theorem.

Theorem 2. *There exists no binary social choice rule which satisfies SPAR and the non-surjectivity of individual inclusion mappings.*

From (5) SPAR implies the *surjectivity of the diagonal mapping*, $(f\circ\Delta)_* \neq 0$, for binary social choice rules. Thus, this theorem is rewritten as follows.

> There exists no binary social choice rule which satisfies the *surjectivity of the diagonal mapping* and the *non-surjectivity of individual inclusion mappings*.

4 Concluding remarks

In Baryshnikov (1997) he said, "the similarities between the two theories, the classical and topological ones, are somewhat more extended than one would expect. The details seem to fit too well to represent just an analogy. I would conjecture that the homological way of proving results in both theories is a 'true' one because of its uniformity and thus can lead to much deeper understanding of the structure of social choice. To understand this structure better we need a much more evolved collection of examples of unifying these two theories and I hope this can and will be done." This paper is an attempt to provide such an example.

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