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Unifying Impossibility Theorems: A Topological Approach*

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In this note a topological approach is developed which permits us to prove Arrow's impossibility theorem in the social choice theory using a topological technique close to that of Chichilnisky. © 1993 Academic Press, Inc.

1. INTRODUCTION

The impossibility theorem of Arrow [1, 4] stays in the center of the modern social choice theory, both for its significance and beauty. However, its proof seems a little irrelevant, as a proof of the fix point theorem without use of algebraic topology, e.g., by means of Sperner's lemma, that is, in a combinatorial way. The beauty and simplicity of Arrow's theorem suggests that it has somewhat more adequate background, which might tie it in with more advanced mathematics and provide possibilities for wider ramifications than those we have now. The title of the paper and the hint above shows in which direction this conjectured background was searched.

There are some impossibility theorems in the social choice theory proven in topological style. This approach was initiated and developed mainly by Chichilnisky (see, e.g., [2, 3]). The bad news is that not only the means were topological, but the goals as well: preferences were defined as smooth functions on $\mathbf{R}^n$, say, nontrivial linear ones, and the aggregation rule was assumed to be continuous in some topology. In general, this theory seems to have more connections with the smooth approach to the equilibrium theory than with the classical social choice theory. However, Chichilnisky's reasonings were as simple as beautiful; that pointed to their fundamentality.

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I briefly recall one of her results (slightly modified). Let the preferences be linear functionals on $\mathbf{R}^n$ having norm one, and an aggregation rule, i.e., a mapping f to the space of preferences $W = S^{n-1}$ from that of profiles $P = W^k$ is given. Impose the following axioms on the mapping: it is continuous, and respects binary unanimity; that is, if an alternative $x \in \mathbf{R}^n$ is preferred to $y \in \mathbf{R}^n$ by all voters, then the aggregating rule estimates x higher than y. We say, that a voter is a manipulator, if for any tuple of preferences of other voters he can choose his preference to produce any a priori given functional as the aggregated preference (that is, a reasonably weakened substitute of dictatorship). Then Chichilnisky's theorem says, that (when $n > 1$) there always exists exactly one manipulator.

The proof consists of two parts. First, one shows that if the voter l is a manipulator then the operator in $(n-1)$-dimensional homologies associated with f maps the lth basis vector of $H_{n-1}(P) = \mathbf{Z}^k$ to the generator of $H_{n-1}(W) = \mathbf{Z}$, otherwise to zero.

The second part of the proof is a simple calculation: the diagonal of P, (i.e., the set of profiles with equal preferences) represents in $(n-1)$-homologies of P the sum of the basic vectors. Its image with respect to f is the generator of $H_{n-1}(W) = \mathbf{Z}$. Thus one and only one of the basic vectors goes to the generator of $H_{n-1}(W) = \mathbf{Z}$; that proves the theorem.

The goal of this note is to show, that combinatorial as it is, Arrow's theorem can be obtained by some arguments of topological nature. To do this we construct some topological spaces naturally associated with the objects of the combinatorial theory. The most surprising fact is how suitable this theory appears to be treated in a topological, or, better, in a homological way.

Let us recall the setup of the Arrow theorem. The initial data are: a natural k—the number of voters; natural $n \geq 3$—the number of candidates. A mapping from the set of profiles, that is, of k-tuples of weak orders on $\{1, 2, \ldots, n\}$ to the set of weak orders itself—the aggregation rule—is given. The following axioms are assumed:

Axiom of Independence of Infeasible Alternatives. The result of comparing of a pair $i, j \in K$ according to the aggregated preference depends only on such results in every weak order of the tuple given.

Unanimity Axiom. If all the voters have the same preference on a pair $\{i, j\}$, then the aggregated rule has to coincide with them on the pair.

We will take the independence axiom as our cornerstone. It says that results of the pairwise comparing are in fact the only thing of interest for us. Another way to say the same is: the preimage (with respect to the mapping defining the aggregation rule) of the set of weak orders with i better than j is a union of products of such sets in the multiples of the set of profiles. Now the intuitive idea is to consider such sets as *open*, and the collection of them as *a covering* of their union. That introduces topology into the setting.

At this point anyone acquainted with the basics of algebraic topology says that the next step is to consider *the nerves* of the coverings in question.

The nerve of a covering of a (topological) space by (open) sets is the simplicial complex, whose vertices one-to-one correspond to the sets of the covering, and a simplex spanned by some vertices lies in the complex if and only if these sets intersect.

Remark 1. Note that we make the picture more complex from the topological point of view: while the topology generated on our finite set of weak orders by the "open" sets was the discrete one, we leave it and arrive at the nerve of a certain covering of this set, whose topology, as we will see, is nontrivial. However, it is the topological nontriviality of the last set that is responsible for the choice paradoxes. Such a nontriviality hidden in the set of orders was noted by D. Saari [5]. However, he did not reveal the topological background of the model, although some of his assertions can be easily reformulated in our terms.

Thus we take two finite sets—of preferences and of profiles—and their special coverings and try to reformulate our initial problem in terms of respective nerves. The topology of the nerve of the covering of preferences we study in Section 3. In Section 4 the topology of the nerve associated with the space of profiles is investigated. It appears that it is close enough to that of the k-fold product of the nerve of preferences. The analogy with Chichilnisky's setup is apparent.

The independence axiom can be easily reformulated in terms of the nerves of the coverings of the spaces of preferences and profiles described above. In fact, the aggregation rule defines the simplicial mapping of these simplicial complexes. The unanimity axiom describes the restriction of the mapping on an analogue of the diagonal. All that is the subject of Section 5.

The dictatorship property also has reformulation as a homological property of the mapping between our simplicial complexes. Thus we arrive in Section 6 with a situation that is completely analogous to that of the topological approach of Chichilnisky.

Remark 2. Such a parallelism cannot be coincidental. I think that it points to the possibility of a unification of the social choice theory on a deeper level. Maybe even more surprising connections are in sight. In fact, all this theory deals with the topology of the permutahedron—the convex hull of the orbit of a generic vector under the action of the group of coordinate permutations. This object is very popular in some rather remote areas of mathematics. This digression needs special consideration, so I will not go into any details here.

I do not define any topological notions here; they all are fairly elementary and standard and can be found in any primer on algebraic topology. All homology and cohomology we use here are singular with $\mathbf{Z}$ as coefficients.

2. Preliminaries

Let $[n]$ denote the set $\{1, 2, \dots, n\}$, interpreted as the set of alternatives to be compared. The set of all weak orders on $[n]$ we will denote by W and that of strict orders (or, equivalently, the set of all permutations of $[n]$) by W_0. Throughout we assume $n \geq 3$. Let $k \geq 2$ be the number of voters. We set $P = W^k$, $P_0 = W_0^k$ to be the set of profiles of preferences on $[n]$.

Let f be an aggregation rule, that is, a mapping of P to W, satisfying the independence of infeasible alternatives and Pareto (binary unanimity) axioms.

Lemma 1. *The image of the restriction of f on P_0 lies in W_0.*

Proof. Let for some $i, j \in [n]$ and some profile $p \in P_0$; we obtain $i =_{f(p)} j$. Then for a profile p' with an element $k \in [n]$ placed between j and i, when j is better with respect to corresponding to the voter's preference, and below j otherwise, we have $i =_{f(p')} j >_{f(p')} k$. For a profile p'' with k between i and j, when i is better, and above j otherwise, we have $k >_{f(p'')} j =_{f(p'')} i$. Thus the result of ranking of i and k depends on the position of j—a contradiction.

Having this lemma in mind we will restrict our further attention to strict orders only, referring to them simply as to orders.

Let $\sigma \in \{+, -\}$; $i, j \in [n]$, $i < j$. Sets of orders with given ranking of elements i, j we will denote as $\mathscr{U}_{ij}^{\sigma}$:

$$\mathscr{U}_{ij}^{+} = \{w \in W_0 \colon i >_w j\}; \qquad \mathscr{U}_{ij}^{-} = \{w \in W_0 \colon i <_w j\}.$$

Obviously, sets $\mathscr{U}_{ij}^{\sigma}$ cover W_0.

Let $\bar{\sigma} = (\sigma_1, \dots, \sigma_k)$ be a vector of signs, $\sigma_l \in \{+, -\}$. We set

$$\mathscr{U}_{ij}^{\bar{\sigma}} = \{p = (w_1, \dots, w_k) \in P_0 \colon w_l \in \mathscr{U}_{ij}^{\sigma_l}\}.$$

Clearly, $\{\mathscr{U}_{ij}^{\bar{\sigma}}\}$ form a cover of P_0.

3. Topology of Preferences

Let N_W be the nerve of the covering of W_0 by the sets $\mathcal{U}_{ij}^{\sigma}$. We will assume henceforth that the sets $\mathcal{U}_{ij}^{\sigma}$ are in an obvious way lexicographically ordered[1] and the N_W is a subcomplex of the $(n(n-1)-1)$-dimensional simplex Δ_{tot}.

Proposition 1. *Dimension of N_W (i.e., the maximal dimension of its simplices) is $(n+1)(n-2)/2$. The simplices of maximal dimension are in one-to-one correspondence with strict orders on $[n]$. The complex N_W coincides with the union of such simplices of maximal dimension.*

Proof. If a simplex in N_W is spanned by more than $n(n-1)/2$ vertices, then at least two vertices are indexed by the same pair $\{i, j\}$. But $\mathcal{U}_{ij}^{+}$ and $\mathcal{U}_{ji}^{-}$ do not intersect. Thus $\dim N_W \leq n(n-1)/2 - 1 = (n+1)(n-2)/2$. Clearly, strict orders lie in the intersection of $(n+1)(n-2)/2$ sets $\mathcal{U}_{ij}^{\sigma}$ with compatible ranking of i and j. It is obvious as well that any nonempty intersection of $(n+1)(n-2)/2$ sets $\mathcal{U}_{ij}$ is a strict order. At last, any set of pairwise comparisons defining acyclic order can be extended to a strict order with pairwise ranking retained. That proves the last statement.

A useful tool to operate with simplices of N_W is to associate to any simplex its graph.

Let δ be an m-dimensional simplex of N_W. Define the graph $g(\delta)$ as oriented with the set of vertices $[n]$ and the set of arrows defined by the vertices of δ: if $\mathcal{U}_{ij}^{+} \in \Delta$, then the arrow from i to j is in $g(\delta)$; if $\mathcal{U}_{ij}^{-} \in \delta$, then that from j to i is in g. Obviously, sets of vertices of the simplices of N_W are in one-to-one correspondence with (simple) oriented graphs on $[n]$ without cycles. An orietantion of δ is canonically defined by the ordering of $\mathcal{U}_{ij}^{\sigma}$'s class of numberings of arrows of $G(\delta)$.

Theorem 1. *The simplicial complex N_W is a homotopy equivalent to the $(n-2)$-dimensional sphere S^{n-2}.*

To prove the theorem we construct a model of N_W, i.e, a manifold, with a covering, which has N_W as its nerve. For such a model we, naturally, will take the n-dimensional Euclidean space $\mathbf{R}^n$ with the deleted diagonal $\Delta = \{(x, \dots, x) \in \mathbf{R}^n\}$, $M = \mathbf{R}^n \setminus \Delta$. Let U_{ij} be the hemispace $\{x_i > x_j\}$. Obviously, the sets $\{U_{ij}\}_{1 \leq i \neq j \leq n}$ cover M. Note that W_0 is naturally embedded in M as the set of vectors whose coordinates form a permuta-

[1]That is, we associate to $\mathcal{U}_{ij}^{+}$ the ordered pair $\{i, j\}$, and to $\mathcal{U}_{ij}^{-}$ the ordered pair $\{j, i\}$, and we order them lexicographically.

tion. Clearly, the coverings of W_0 by $\mathscr{U}_{ij}^{\sigma}$'s and U_{ij}'s coincide; thus N_W is a subcomplex of the nerve N_M of the covering of M by U_{ij}'s.

LEMMA 2. *These complexes coincide*: $N_W = N_M$.

The proof is obvious; any order on a finite set can be represented by a function.

Proof of the Theorem 1. All the sets U_{ij} are *convex* subsets of Euclidean space; thus their intersections are convex as well, and the covering is fine. That is, all the sets and their intersections are contractible. It follows that the nerve of the covering, which is N_W, has the same homotopy type as M itself, which obviously has the homotopy type of the $(n-2)$-dimensional sphere S^{n-2}.

There is a very important class of cycles representing the generator of $(n-2)$th homology group of N_W (isomorphic to $\mathbf{Z}$). Namely, let us choose a simple cycle of length n on $[n]$, considered as an unoriented graph. Any orientation of its edges gives us a subset of the set of vertices of the "large" simplex Δ_{tot}, i.e., its subsimplex spanned by vertices corresponding to arrows. Denote the oriented graph (whose unoriented support is our cycle) by g, and the corresponding subsimplex of the total simplex as $\delta(g)$ (note that the orientation of this simplex is canonically defined through the ordering of vertices of the total simplex).

Let the chain $h(g)$ be the boundary of $\delta(g)$. Obviously, $h(g)$ is a cycle.

PROPOSITION 2. *The cycle $h(g)$ lies in the group of $(n-2)$-dimensional cycles of N_W. If g is an oriented cycle (remember, g is an oriented graph), then $h(g)$ represents a generator of $H_{n-2}(N_W)$; otherwise it is zero.*

Proof. The chain $\partial\delta(g)$ is a sum of $(n-2)$-dimensional simplices spanned by all but one of the arrows of g. These arrows obviously define some acyclic relation on $[n]$, which can be extended to some strict order. This strict order defines some simplex of N_W (of maximal dimension), which has the simplex under consideration as a face. Thus all simplices of the cycle $c(g)$ are in N_W. If g is not an oriented cycle, then it can be extended to a strict order as well, and the simplex spanned by arrows of g lies in N_W, annihilating $h(g)$ in the homologies. If g is an oriented cycle, then the halfspaces corresponding to its arrows cover $\mathbf{R}^n \setminus \Delta$. Thus the embedding of the (homotopic) sphere $h(g)$ into N_W is the homotopy equivalence, and $h(g)$ represents a generator of H_{n-2}.

4. TOPOLOGY OF SPACE OF PROFILES

As we know, sets $\mathscr{U}_{ij}^{\bar{\sigma}}$ form a covering of P_0. Denote the nerve of the covering by N_P.

PROPOSITION 3. *Dimension of N_P (i.e., the maximal dimension of its simplices) equals the dimension of N_W. The simplices of the maximal dimension are in one-to-one correspondence with k-tuples of strict orders on $[n]$. The complex N_P coincides with the union of closures of its simplices of maximal dimension.*

Proof. Any simplex with the number of vertices larger than $n(n-1)/2$ would have at least two vertices with the same pair $\{i, j\}$ of indices. Corresponding sets either coincide or are disjoint. Other statements are obvious.

To study the topology of N_P we will apply once again the approach of the previous section. Let $(\mathbf{R}^n)^k = \mathbf{R}^{nk}$ be the space of k-tuples of n vectors. As before, we use an identification of orders on finite set $[n]$ with $\mathbf{R}$-valued functions on $[n]$.[2] Now we have to describe the topology of the model M_P of N_P which is the union of sets,

$$\left\{(\mathbf{x}^1, \dots, \mathbf{x}^k), \mathbf{x}^l \in \mathbf{R}^n\colon x_i^l > x_j^l \text{ if } \sigma_l = +, \text{ and } x_i^l < x_j^l \text{ if } \sigma_l = -\right\},$$

$\bar{\sigma} = (\sigma_1, \dots, \sigma_n) \in \{+, -\}^n$, $i < j \in [n]$, for which we will keep notation $\mathcal{U}_{ij}^{\bar{\sigma}}$. The covering of M_P by the sets $\mathcal{U}_{ij}^{\bar{\sigma}}$ is fine, because they are convex subsets of $\mathbf{R}^{nk}$ and the nerve of this covering coincides with N_P.

The topology of M_P is rather complicated. But at least in the dimensions we need its homology behaves exactly as one could wish, i.e., as that of a direct product.

PROPOSITION 4. *The (singular, with coefficients in $\mathbf{Z}$) cohomology groups of M_P are zeros in positive dimensions up to $n-2$, and $H^{n-2}(M_P) = \mathbf{Z}^k$.*

Proof. For a pair $\{i, j\}$ take the union of $\mathcal{U}_{ij}^{\bar{\sigma}}$ over all $\bar{\sigma}$'s. The result is the product of k items of $\mathbf{R}^n$ with deleted hyperplane $\{x_i = x_j\}$, i.e., $\mathbf{R}^{nk}$ without the cross C_{ij} which is the union of ij-diagonals $\{x_i = x_j\}$ in its multiples. The union of all such $\mathbf{R}^{nk}$'s with deleted crosses over all pairs $\{i,j\}$ can be represented as $\mathbf{R}^{nk}$ without the union of intersections over all pairs $\{i, j\}$ of diagonals $\{x_i = x_j\}$ in (possibly) different multiples. The count of the dimensions shows that such intersections have codimension $n-1$ or more and that the only hyperplanes of the codimension $(n-1)$ are the diagonals $\{x_1 = x_2 = \cdots = x_n\}$ in multiples $\mathbf{R}^n$ of $\mathbf{R}^{nk}$. That, obviously, proves the proposition.

[2]Of course, there are a lot of functions corresponding to the same order, but by restricting our attention on some special coverings and having proved that our problems can be encoded in terms of these coverings, we can hope that the topology of the nerve of the covering, reflecting in turn the topology of our manifolds, faithfully represents the combinatorial features of our initial problem.

Thus homology and cohomology groups of N_W in dimensions up to $n-2$ are as in the k-fold product of N_W. The mapping of vertices of N_P to these of N_W, sending the tuple of comparisons to its lth component can be extended to the mapping of the simplicial complexes p_l: $N_P \to N_W$.

PROPOSITION 5. *Let c be a generator of $H^{n-2}(N_W)$. Then elements $p_l^*(c)$ form the basis of $H^{n-2}(N_P) = \mathbf{Z}^k$. (Here p_l^* are linear operators in cohomologies, induced from p_l).*

Proof. We will construct in $(n-2)$-homologies of N_P a basis which will consist of k cycles h_l, which p_m project into the respective generators of $H_{n-2}(N_W)$ when $k = m$, and to zero otherwise. Let $\mathbf{g} = (g_1, g_2, \ldots, g_k)$ be a k-tuple of oriented graph on $[n]$ with the same unoriented support, which is a cycle of length n; for definitivity we take the cycle $1-2-\cdots-n-1$. Such a tuple defines the set of n vertices of N_P in obvious way: $\mathscr{U}_{ii+1}^{\bar{\sigma}}$ has the sign σ_l as directed by the arrow between i and $i+1$ in the respective graph g_l. Any $n-1$ of these vertices define a simplex in N_P; their sum with the respective signs is a cycle. We will denote it as $h(\mathbf{g})$. This cycle upon projection p_l goes to the cycle $h(g_l)$ of N_W. If we take k-tuple of graphs $\mathbf{g}$ having the oriented cycle $1 \to 2 \to \cdots \to n \to 1$ as its lth graph and acyclic graphs on other places, we will obtain the basis in question. The pairing of h_l and $p_l^*(c)$ gives, obviously, δ_{lm}. Thus $\{p_l^*(c)\}_{1\le l\le k}$ is the basis in $H^{n-2}(N_P)$ (and $\{h_l\}_{1\le l\le k}$ is the basis in $H_{n-2}(N_P)$).

Let D be the mapping of N_W to N_P, which sends $\mathscr{U}_{ij}^{\sigma}$ to $\mathscr{U}_{ij}^{\bar{\sigma}}$, where $\bar{\sigma}$ is the replica-vector of σ. Obviously, this mapping can be extended to that of simplicial complexes. We will call it diagonal embedding.

5. TOPOLOGY OF AGGREGATION RULE

Remember that the aggregation rule is a mapping $f: P \to W$. Clearly, the axiom of independence of infeasible preferences is equivalent to the fact that the f-image of the subset $\mathscr{U}_{ij}^{\bar{\sigma}}$ is in one subset $\mathscr{U}_{ij}^{\sigma}$, which defines the mapping from N_P to N_W, for which we will keep the notation f. Thus the following diagram of simplicial mappings is defined:

$$\begin{tikzcd} N_W \arrow[r, "D"] & N_P \arrow[r, "f"] \arrow[dl, "p_1"'] \arrow[dr, "p_k"] & N_W \\ N_W & \cdots & N_W \end{tikzcd}$$

PROPOSITION 6. *All composite mappings from N_W to itself in this diagram are identical.*

The only thing to note here is that the equality of $D \circ f = id_{N_W}$ is equivalent to the binary unanimity (or Pareto) axiom.

Now we will consider how dictatorship reflects in homologies. Let $\{h_l\}_{l=1,\dots,k}$, $h_l \in H_{n-2}(N_P)$, be the basis of homologies of N_P, constructed above; in the proof of Proposition 5, c is the generator of $H^{n-2}(N_W)$.

PROPOSITION 7. *If voter l is a dictator, then $(h_l, f^*(c)) = 1$; otherwise it is* 0.

Proof. Let $\mathbf{g}$ be the k-tuple of oriented graphs with the same unoriented support $1 - 2 - \cdots - n - 1$, whose associated cycle $h(\mathbf{g})$ represents h_l in $H_{n-2}(N_P)$. Clearly, the image of $h(\mathbf{g})$ with respect to f is again a cycle in N_W, defined by an oriented graph g with the same unoriented support. Let g_l be the cycle $1 \to 2 \to \cdots \to n \to 1$. If g is not an oriented cycle, then the images of all simplices of $h(\mathbf{g})$ have common vertex $\mathscr{U}_{12}^{\sigma}$, say $\mathscr{U}_{12}^{+}$. It follows that f maps a profile with lth preference $2 \to \cdots \to n \to 1$ to a preference with 1 over 2, and l is not a dictator. If the common vertex is $\mathscr{U}_{12}^{-}$, then l is not a dictator because the aggregated rule contradicts its preferences on $\{1, 2\}$.

Of course, that g is not an oriented cycle is equivalent to $(f_*(h(\mathbf{g})), c) = (f_*(h_l), c) = (h_l, f^*(c)) = 0$, where the parentheses denote pairing between cohomologies and homologies on N_P.

Inversely, let $f_*(h_l)$ be a generator of $H_{n-2}(N_W)$. That means that g is an oriented cycle. If we will vary all but lth components of $\mathbf{g}$ in such a way that graphs g_m, $m \neq l$ are not oriented cycles and graph g remains the same. Indeed, all mentioned changes can be done sequentially by changing the orientation of one arrow in one component. That can lead only to a change of the orientation of only one arrow of g. But g have to remain the oriented cycle; thus it is rigid. By the described changes one can achieve any tuples of orientations of arrows of g_m, $m \neq l$ between 1 and 2; the resulting preference will be the same. Particularly, if all the orientations agree, we obtain that the aggregated preference on $\{1, 2\}$ coincides with that of the lth voter. The cycle g_l representing the homological generator can be chosen to include any arrow $i \to j$. Thus l is a dictator.

6. EASY CALCULATIONS

Let us axiomatize the resulting picture we have found in the previous section. Let A and B be two simplicial complexes, such that in some dimension m we have $H^m(B) = \oplus_{i=1}^k H^m(A)$, and the commutative diagram of simplicial mapping is

$$\begin{tikzcd} & A \arrow[r, "D"] & B \arrow[r, "f"] \arrow[dl, "p_1"'] \arrow[dr, "p_k"] & A \\ & A & \cdots & A \end{tikzcd}$$

such that all composite mappings from A to itself in this diagram are identical and that all $(p_k)_*$ act in the mth homologies as projections on direct factors in the above decomposition. We will call such data *purely separated* (*in dimension* m).

DEFINITION. Let we have purely separated the data as above. The index l, $l = 1, \ldots, k$, is said to be a homological dictator if the composite mapping

$$H_m(A) \xrightarrow{(i_l)_*} H_m(B) \xrightarrow{f_*} H_m(A)$$

is an isomorphism and homologically irrelevant if it is trivial.

Now what is proved in Section 4 is the pure separateness of Arrow's data (in our setup) and in Section 5 is that any dictator (in usual sense) is a homological dictator and that any non-dictator is homologically irrelevant.
Exactly in the same way one sees that in Chichilnisky's setup the data are purely separated and that a voter is a manipulator iff he is a homological dictator, and he is homologically irrelevant otherwise. Now both Arrow's and Chichilnisky's impossibility theorems follows from a simple theorem.

THEOREM 2. *If we have purely separated* (*in dimension* m) *data such that the mth homological group of A contains free part* (*that is*, $\mathbf{Z}$). *Assume that any index* $l \in \{1, \ldots, k\}$ *is either a homological dictator or homologically irrelevant. Then there exists exactly one homological dictator.*

For the proof we need one more lemma.

LEMMA 3. *The homological embedding* D_* *of* $H_m(A)$ *into* $H_*(B)$ *is the diagonal one*; *that is, for any* $h \in H_m(A)$ *we have* $D_*(h) = (h, \ldots, h) \in H_*(B)$.

The proof of the lemma reduces to the definition of pure separatedness.

Proof of the theorem. Let $h \in H_m(A)$ is a free generator. According to the previous lemma, the composite mapping $f_* \circ D_*$ takes h into $\sum_{l=1}^k d_l h$, where $d_l = 1$ if l is homological dictator, and zero otherwise. As it has to be identical, the sum equals one, thus proving the theorem.

Thus a homological dictator always exists and is unique.

7. Concluding Remarks

7.1. *Original Arrow's Theorem*

As it was published, the impossibility theorem of Arrow needs less. Specifically, it is assumed, that f is defined not on the whole P, or P_0, but on a subset, such that for any triple of candidates the restriction of admissible profiles on this triple is full; i.e., it contains all possible profiles of preferences on the triple. Note, that in this case we can again apply our technique. The only thing that remains to prove is that dictators for different triples are the same voter. That again can be done through topology. Specifically, one can show that in the two-skeleton of N_P (which is responsible for $n - 2 = 3 - 2 =$ one-dimensional (co)homology), respective generators for different triples are homological; thus the dictator for all triples is the same.

7.2. *Domain Conditions*

The usual approach to "resolve the independence paradox" is to introduce a domain condition, i.e., to reduce in some way the set of admissible preferences. The most simple and known approach in "discrete" social choice theory is "single-peakedness": for an ordering of $[n]$ (say, $1 \to 2 \to \cdots \to n$) only such orders are allowed for which taking sequential alternatives $1, 2, \ldots, n$ one meets only one local maximum, that is, orders having the form $1 \leftarrow 2 \leftarrow \cdots \leftarrow i \to \cdots \to n$.

On the topological side, Chichilnisky and Heal [3] have shown that the proper domain restriction is contractibility of the space of preferences (that is, its homotopy equivalence to the point).

Our considerations suggest that "discrete" domain restrictions have to be formulated in terms of the topology of N_W with some deleted simplices (forbidden orders) of maximal dimension. Ideally were the killing of all homology groups. Moreover, as the presence of a dictator on any subset of candidates is not desirous, the respective homology groups of subcomplexes of N_W corresponding to these subsets have to be zero as well.

An intriguing fact is that this goal is achieved for "single-peaked" orders. For $n = 3$ the vanishing of homologies and single peakedness are equivalent. I hope to return to the question in subsequent publications.

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