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Monotonicity of single-seat preferential election rules

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Abstract

Various properties of preferential election rules are described, including nine forms of monotonicity. It is shown that Condorcet's principle is incompatible with many of them. Some progress is made towards the task of determining all maximal mutually compatible subsets of these properties. To that end, a survey is given of the monotonicity properties of many known single-seat preferential election rules, and four new rules are described, including one that is offered as a more monotonic practical alternative to the Alternative Vote.

Keywords: Monotonicity; Preferential election rule; Preferential voting rule; Electoral system; Voting paradox; Condorcet principle; Droop proportionality criterion; Alternative vote; Single transferable vote; STV

1. Introduction

It is well known [4, 10] that the Single Transferable Vote (STV) fails various tests of monotonicity. A major unsolved problem is whether there exist rules that retain the important political features of STV and are also more monotonic. Because this is such a difficult problem in general, I concentrate here on the special case of single-seat elections, in which STV reduces to AV, the Alternative Vote. However, the basic definitions are given (in Section 2) and the properties are described (in Section 3) in a form that applies equally well to multi-seat elections. Two impossibility theorems are proved in Section 4, which show that various properties, including Condorcet's principle, are incompatible with many forms of monotonicity. The monotonicity properties of many single-seat election rules are surveyed in Sections 5–7: some known non-Condorcet rules in Section 5, some Condorcet-type rules (including two new ones, PMM and MMG) in Section 6, and two new rules (QLTD and DAC) in Section 7. DAC is proposed as a more monotonic practical alternative to the Alternative Vote. A summary of properties is given in Table 1.

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2. Definitions

2.1. Terminology and axioms for election rules

Candidates will be denoted by lower-case letters $a, b, c, \ldots$. Each voter casts a *ballot* containing a *preference listing* of the candidates, which is written as (for example) abc, to denote that the voter places a first, b second and c third, with no fourth choice being expressed. A preference listing is *complete* if it includes all candidates, and *truncated* otherwise. A *profile* is a weighted family of preference listings, such as might represent the ballots cast in an election. Profiles are represented as for Elections 1 and 2 below, indicating either the proportion, or the absolute number, of ballots of each type cast.

Election 1 (1 seat)		Election 2 (2 seats)			
ab	0.17	a	9	ea	4
ac	0.16	b	9	eb	4
bac	0.33	c	10	c	1
cb	0.34	d	10	fd	1
				fe	6

In an election to fill s seats from n candidates, an *outcome* is a set of s candidates; so there are $\binom{n}{s}$ different outcomes. As in [20], a (preferential) *election rule* (for filling s seats) is a function that associates with each profile a probability space on the set of corresponding outcomes. The 'normal' situation is that all outcomes are given probability 0 except for one, which has probability 1; if anything else happens, then we say that the result is a *tie* between all the outcomes that have non-zero probability. For example, if AV is used for Election 1 above, then it elects a with probability $\frac{1}{2}$, and b and c with probability $\frac{1}{4}$ each; and if STV is used for Election 2, then it elects $\{a, b\}$ with probability $\frac{1}{2}$ and $\{c, e\}$ and $\{d, e\}$ with probability $\frac{1}{4}$ each.

This definition implies that every election rule is *anonymous*, meaning that the result depends only on the number of ballots of each type in the profile. It is *neutral* (hence, being anonymous, it is *symmetric*) if, whenever a permutation is applied to the names of the candidates on all the ballots, then the same permutation is applied to the result. It is *homogeneous* if the result depends only on the *proportion* of ballots of each type, not on their absolute number. And it is *discriminating* if, for every possible set of preference listings, the proportion of profiles that give rise to a tie, out of all profiles that include only preference listings from that set, tends to zero as the number of voters in the profile tends to infinity. (This is a rather stronger form of discrimination than is usually imposed, but we shall need it in the proof of Theorem 3. The imposition of discrimination in *any* form rules out systems that use random selection as an intrinsic feature and not just as a last resort; such systems are of great interest

[1, 8] but they are outside the scope of this work.) A *proper* election rule is one that is symmetric, homogeneous and discriminating. Henceforth we shall consider only proper election rules.

Note that election rules involving transfers of fractional votes are seldom homogeneous in practice, because replicating each ballot a large number of times will reduce the effect of any rounding errors. We shall count an election rule as *proper* if an idealized version of it is proper, even though a specific implementation of it may not be.

2.2. Notation for profiles

We shall always denote the set of candidates by C, the number of seats (= the number of candidates to be elected) by s, the total number of votes (= ballots in the profile) by v, the number of ballots containing candidate x by $v(x)$, and the number with x in ith place by $v_i(x)$. The *Droop quota* is $v/(s+1)$. A voter, ballot or preference listing *prefers* x to y if he, she or it lists x above (before) y, or lists x but not y. We write $S(X)$ for the set of voters who are *solidly committed* to a set $X \subseteq C$, that is, who prefer every candidate in X to every candidate not in X (see [7]). We write $E(X)$ for the set of candidates in X who are elected (by the election rule currently under consideration); $P(|F|E(X)| \geqslant 1)$ is the probability that this number is positive, and $P_{\mathrm{E}}(x) = P(|E(\{x\})| = 1)$ is the probability that candidate x is elected.

We write $g(x, y)$ for x's *gross majority* over y, that is, the number of voters who prefer x to y. Then $n(x, y) = g(x, y) - g(y, x)$ is x's *net majority* over y. We define the *minimum gross score* mings(x), *minimum net score* minns(x) and *maximum net antiscore* maxna(x) of candidate x by

$$\mathrm{mings}(x) = \min_{y \in C\setminus\{x\}} g(x, y), \qquad \mathrm{minns}(x) = \min_{y \in C\setminus\{x\}} n(x, y),$$

$$\mathrm{maxna}(x) = \max_{y \in C\setminus\{x\}} n(y, x).$$

Clearly $\mathrm{minns}(x) = -\mathrm{maxna}(x)$ and, in the absence of truncated preference listings, $\mathrm{mings}(x) = \frac{1}{2}(v + \mathrm{minns}(x))$; but there is no direct connection between mings(x) and minns(x) in general.

3. Properties of election rules

It is convenient to divide these into *global* or *absolute* properties on the one hand, and *local* or *relative* properties on the other. The former say something about the result of applying an election rule to a single profile, whereas the latter say how the result should (or should not) change when certain changes are made to the profile.

3.1. Global or absolute properties

The essential feature of STV, which makes it a system of proportional representation (within each constituency), is what I call the *Droop proportionality criterion* or DPC. The single-seat version of DPC is what I call MAJORITY. PLURALITY is a rather weak property that surely must hold in any real election.

- DPC. If, for some integer k and set $X \subseteq C$ satisfying $0 < k \leqslant |X|$, $|S(X)|$ exceeds k Droop quotas, then $|E(X)| \geqslant k$ (with probability 1).
- MAJORITY. If $|S(X)| > \frac{1}{2}v$, then $|E(X)| \geqslant 1$.
- PLURALITY. If $v(x) < v_1(y)$, then $P_E(x) < P_E(y)$.

We say that x *beats* y or *ties with* y (in pairwise comparisons) if $n(x, y)$ (defined in Section 2.2) satisfies $n(x, y) > 0$ or $= 0$, respectively. A *Condorcet winner* (resp. *Condorcet non-loser*) [5] is a candidate who beats (resp., beats or ties with) every other candidate in pairwise comparisons; note that all Condorcet non-losers (if any) must tie with each other. The *Condorcet top tier* is the smallest non-empty set $T \subseteq C$ such that every candidate in T beats every candidate (if any) outside T. (This concept was apparently first introduced by Nanson [15].) Let W (resp. L) be the set of Condorcet winners (non-losers). Then $|W| \leqslant 1$ and $L \subseteq T$, and if $|W| = 1$ then $W = L = T$.

Condorcet's principle and the two strengthenings of it given below were formulated originally for single-seat elections in which every voter provides a complete preference listing; but I have reworded them here so that they make sense (even if they are not necessarily sensible) for all preferential elections.

- CONDORCET [5]. If there is a Condorcet winner w, then w should be elected.
- SMITH–CONDORCET [17]. $|E(T)| \geqslant 1$.
- EXCLUSIVE-CONDORCET (see [9]). If $L \neq \emptyset$ then $|E(L)| \geqslant 1$.

Note that SMITH–CONDORCET and EXCLUSIVE-CONDORCET both imply CONDORCET, and SMITH–CONDORCET also implies MAJORITY. It is easy to see that, in multi-seat elections, DPC and CONDORCET are mutually incompatible. Many authors have found CONDORCET an attractive principle for single-seat elections, although others [9] have found it less plausible, and we shall see in Theorem 2 that it conflicts with many monotonicity properties.

3.2. Local or relative properties

We shall say that a candidate x is *helped* or *harmed* by a change in the profile if the result is, respectively, to increase or to decrease $P_E(x)$. The following two properties are well known to hold for STV.

- LATER-NO-HELP. Adding a later preference to a ballot should not help any candidate already listed.
- LATER-NO-HARM. Adding a later preference to a ballot should not harm any candidate already listed.

Next we come to the different versions of monotonicity. The basic theme is that a candidate x should not be harmed by a change in the profile that appears to give more support to x; but one gets different flavours of monotonicity if one specifies different ways in which the profile might be changed.

MONOTONICITY. A candidate x should not be harmed if:

- (MONO-RAISE) x is raised on some ballots without changing the orders of the other candidates;
- (MONO-RAISE-DELETE) x is raised on some ballots and all candidates now below x on those ballots are deleted from them;
- (MONO-RAISE-RANDOM) x is raised on some ballots and the positions now below x on those ballots are filled (or left vacant) in any way that results in a valid ballot;
- (MONO-APPEND) x is added at the end of some ballots that did not previously contain x;
- (MONO-SUB-PLUMP) some ballots that do not have x top are replaced by ballots that have x top with no second choice;
- (MONO-SUB-TOP) some ballots that do not have x top are replaced by ballots that have x top (and are otherwise arbitrary);
- (MONO-ADD-PLUMP) further ballots are added that have x top with no second choice;
- (MONO-ADD-TOP) further ballots are added that have x top (and are otherwise arbitrary);
- (MONO-REMOVE-BOTTOM) some ballots are removed, all of which have x bottom, below all other candidates.

There is also the following property, which is not strictly a form of monotonicity but is very close to it (and is again reworded here for multi-seat elections).

- PARTICIPATION [14]. For $X \subseteq C$, the addition of further ballots that are solidly committed to X should not reduce $P(|E(X)| \geqslant 1)$.

All the monotonicity properties seem superficially desirable, except that, as explained in [20], I do not think that MONO-REMOVE-BOTTOM is desirable in multi-seat elections. Of course, any of these properties may be undesirable in practice if it turns out to have undesirable consequences. MONO-RAISE-RANDOM, in particular, is very restrictive. Among many questions that I cannot answer is whether there exists an election rule that satisfies both MONO-RAISE-RANDOM and MAJORITY; if not, then I would certainly regard MAJORITY as the more important property to preserve.

Theorem 1. *The following implications hold*:

(a) MONO-RAISE-RANDOM $\Rightarrow$ (MONO-RAISE *and* MONO-RAISE-DELETE);
(b) (MONO-RAISE *and* LATER-NO-HELP) $\Rightarrow$ MONO-RAISE-DELETE;
(c) (MONO-RAISE-DELETE *and* LATER-NO-HARM) $\Rightarrow$ MONO-RAISE-RANDOM;
(d) MONO-SUB-TOP $\Rightarrow$ MONO-SUB-PLUMP;
(e) (MONO-SUB-PLUMP *and* LATER-NO-HARM) $\Rightarrow$ MONO-SUB-TOP;
(f) (MONO-APPEND *and* MONO-RAISE-DELETE) $\Rightarrow$ MONO-SUB-PLUMP;
(g) (MONO-APPEND *and* MONO-RAISE-RANDOM) $\Rightarrow$ MONO-SUB-TOP;
(h) MONO-ADD-TOP $\Rightarrow$ MONO-ADD-PLUMP;
(i) (MONO-ADD-PLUMP *and* LATER-NO-HARM) $\Rightarrow$ MONO-ADD-TOP;
(j) PARTICIPATION $\Rightarrow$ MONO-ADD-TOP.

Moreover, in single-seat elections,

(k) PARTICIPATION ⇒ MONO–REMOVE-BOTTOM.

Also, if truncated preference listings are not allowed, then

(l) MONO–RAISE-RANDOM ⇒ MONO–SUB-TOP.

Proof. These are all straightforward. □

4. Impossibility theorems

We shall prove two multi-part impossibility theorems. Note that if a number of properties are compatible in general, then in particular they must be compatible for single-seat elections, and so it suffices to prove their incompatibility in this case. We shall assume throughout that all election rules are proper.

Theorem 2. (a) *Even if truncated preference listings are not allowed,* CONDORCET *is incompatible with* PARTICIPATION, MONO–RAISE-RANDOM *and* MONO–SUB-TOP.

(b) *In general,* CONDORCET *is incompatible with* LATER-NO-HELP, LATER-NO-HARM, MONO–RAISE-DELETE, MONO–SUB-PLUMP *and, in the presence of* PLURALITY, MONO–ADD-TOP.

(c) *There is no election rule that satisfies* LATER-NO-HELP *and* LATER-NO-HARM, *and that also satisfies* CONDORCET *whenever there are no truncated preference listings.*

Proof. For the incompatibility of CONDORCET with PARTICIPATION, see [14].

Election 3 (1 seat)				Election 4 (1 seat)	
abc	3	*acb*	2	*ab*	11
bca	3	*bac*	2	*b*	7
cab	3	*cba*	2	*c*	12

Consider Election 3. By symmetry, the result must be a 3-way tie; but, by the axiom of discrimination, there must be a profile P arbitrarily close to this (in the proportions of ballots of each type) that does not yield a tie. Without loss of generality, suppose a is elected in P. But c becomes the Condorcet winner, and so must be elected by CONDORCET, if half the *bac* ballots in P are replaced by *abc* and half by *acb* (contrary to MONO–RAISE-RANDOM and MONO–SUB-TOP), or if all the *abc* ballots are replaced by a (contrary to LATER-NO-HELP), or if all the *bac* ballots are replaced by a (contrary to MONO–RAISE-DELETE and MONO–SUB-PLUMP), or if all the *abc* ballots are replaced by *acb* (contrary to LATER-NO-HELP and LATER-NO-HARM together). This proves (a), (c) and three parts of (b).

Suppose we modify the profile in Election 3 by deleting the second and third choices from all the *abc*, *bca* and *cab* ballots. Again, there must be a profile P' arbitrarily close to the modified profile that does not yield a tie, and we may suppose w.l.o.g. that a is elected in P'. But b becomes the Condorcet winner if we replace the a ballots in P' by abc, contrary to LATER-NO-HARM.

Finally, consider Election 4. Again, even if this yields a tie, there must be a profile P'' arbitrarily close to it that does not. By PLURALITY, a cannot be elected in P'' (because of c); by CONDORCET and MONO-ADD-TOP, b cannot be elected, because adding two ba ballots would make a the Condorcet winner; and similarly c cannot be elected, because adding five cb ballots would make b the Condorcet winner. This contradiction completes the proof. □

We shall see in later examples that CONDORCET (with or without PLURALITY) is compatible with all forms of monotonicity not specifically ruled out by Theorem 2. I interpret Theorem 2(b) as saying that CONDORCET is not desirable when truncated preference listings are allowed.

Theorem 3. *A proper election rule that satisfies* MAJORITY, LATER-NO-HELP *and* LATER-NO-HARM *cannot satisfy any of the following*:

(i) MONO-SUB-PLUMP *or* MONO-SUB-TOP;

(ii) MONO-RAISE, MONO-RAISE-DELETE *or* MONO-RAISE-RANDOM;

(iii) MONO-REMOVE-BOTTOM *or* PARTICIPATION.

Proof. We again consider only single-seat elections. By Theorem 1, in the presence of LATER-NO-HELP and LATER-NO-HARM, the two MONO-SUB properties are equivalent, as are the three MONO-RAISE properties. Also (for single-seat elections) PARTICIPATION implies MONO-REMOVE-BOTTOM. Thus it suffices to prove the incompatibility using the first property mentioned in each of (i)–(iii). We write $P(X \to x)$ for the probability that x is elected (by the rule currently under consideration) in profile X.

(i) This is a stronger version of the theorem (and proof) from [18]. Suppose an election rule satisfies MAJORITY, LATER-NO-HELP, LATER-NO-HARM and MONO-SUB-PLUMP. Consider the following profiles.

A_0		A_1	A_2		A_3	A_4		A_5	A_6	
a	$0.335-\frac{1}{2}\varepsilon$	a	ab	$0.34+\delta$	a	ab	$0.34+\delta$	ab	ab	$0.3+\varepsilon$
b	$0.33+\varepsilon$	b	b	$0.33+\varepsilon$	b	b	$0.3+\varepsilon$	b	ba	$0.3+\varepsilon$
c	$0.335-\frac{1}{2}\varepsilon$	c	c	$0.33-\delta-\varepsilon$	c	c	$0.36-\delta-\varepsilon$	c	c	$0.4-2\varepsilon$

By the axiom of discrimination, we can choose δ and ε so that $|\delta|+|\varepsilon|<0.001$ and neither of profiles A_1 and A_3 results in a tie. Since $P(A_0 \to a) = P(A_0 \to c)$ by symmetry, it follows by MONO-SUB-PLUMP that $P(A_1 \to c) \leqslant P(A_0 \to c) \leqslant \frac{1}{2}$, whence $P(A_1 \to c) = 0$. By similar arguments, $P(A_3 \to a) = P(A_1 \to b) = 0$, and so $P(A_1 \to a) = 1$. Now

$P(A_2 \to a) \geqslant P(A_1 \to a) = 1$ by LATER-NO-HARM,

$P(A_4 \to b) \leqslant P(A_2 \to b) = 0$ by MONO-SUB-PLUMP,

$$P(A_4 \to a) \leqslant P(A_3 \to a) = 0 \quad \text{by LATER-NO-HELP},$$

$$P(A_5 \to c) \geqslant P(A_4 \to c) = 1 \quad \text{by MONO-SUB-PLUMP},$$

and

$$P(A_6 \to b) \leqslant P(A_5 \to b) = 0 \quad \text{by LATER-NO-HELP}.$$

However, $P(A_6 \to b) = P(A_6 \to a) = \frac{1}{2}$ by MAJORITY and symmetry, and this contradiction proves (i).

(ii) I am indebted to B.J. Tarlow for the central part of the argument in (ii) and (iii). Consider the following profiles.

B_0	B_1	B_2	B_3	B_4	B_5	B_6	
ac	ac	acb	abc	a	acb	acb	$0.3+\delta$
b	b	b	bac	bac	bac	bac	$0.3+\varepsilon$
ca	c	cba	c	c	c	cba	$0.4-\delta-\varepsilon$

	C_0		C_1	C_2
ac	$0.35-\frac{1}{2}\varepsilon$	acb	$0.3\dot{3}$	$0.3\dot{3}$
b	$0.3+\varepsilon$	bac	$0.3+\varepsilon$	$0.3\dot{3}$
ca	$0.35-\frac{1}{2}\varepsilon$	cba	$0.3\dot{6}-\varepsilon$	$0.3\dot{3}$

Suppose an election rule satisfies MAJORITY, LATER-NO-HELP, LATER-NO-HARM and MONO-RAISE. Let B_i' denote the profile obtained from B_i by interchanging δ and ε. By the axiom of discrimination, we can choose δ and ε so that $|\delta| + |\varepsilon| < 0.001$ and none of profiles B_1, B_1', B_3 and B_3' result in a tie. Since $P(B_3 \to a) + P(B_3' \to a) = 1$ by MAJORITY and symmetry, we may suppose by interchanging δ and ε if necessary that $P(B_3 \to a) = 1$. Since $P(C_0 \to a) = P(C_0 \to c) = \frac{1}{2}$ by MAJORITY and symmetry,

$$P(B_1 \to c) \geqslant P(B_0 \to c) \geqslant P(C_0 \to c) = \tfrac{1}{2} \quad \text{by LATER-NO-HELP and MONO-RAISE},$$

and so $P(B_1 \to c) = 1$. Also

$$P(B_5 \to a) \geqslant P(B_4 \to a) \geqslant P(B_3 \to a) = 1 \quad \text{by LATER-NO-HARM and LATER-NO-HELP}.$$

Now

$$P(B_2 \to c) \geqslant P(B_1 \to c) = 1 \quad \text{by LATER-NO-HARM},$$

$$P(B_6 \to b) \leqslant P(B_2 \to b) = 0 \quad \text{by LATER-NO-HELP},$$

$$P(B_6 \to c) \leqslant P(B_5 \to c) = 0 \quad \text{by LATER-NO-HELP},$$

$$P(C_1 \to a) \geqslant P(B_6 \to a) = 1 \quad \text{by MONO-RAISE},$$

and

$$P(C_2 \to c) \leqslant P(C_1 \to c) = 0 \quad \text{by MONO-RAISE}.$$

However, $P(C_2 \to c) = P(C_2 \to a) = P(C_2 \to b) = \frac{1}{3}$ by symmetry, and this contradiction proves (ii).

(iii) The argument here is very similar to that of (ii), but with different start and finish. Let α, β and γ be positive integers, let $\mu := \max(\alpha, \beta)$, and suppose $\gamma > \mu$ and $2\alpha > \frac{1}{2}(\alpha + \beta + \gamma)$. Consider the following profiles.

D_0		D_1		D_2			D_3	D_4	D_5
ac	α	ac	α	ac	α	acb	α	α	μ
b	β	b	β	b	β	bac	β	β	μ
ca	α	ca	α	c	γ	cba	γ	μ	μ
		cba	$\gamma - \alpha$						

Suppose an election rule satisfies MAJORITY, LATER-NO-HELP, LATER-NO-HARM and MONO-REMOVE-BOTTOM. By majority and symmetry, $P(D_0 \to a) = P(D_0 \to c) = \frac{1}{2}$. Therefore

$$P(D_1 \to a) \leqslant P(D_0 \to a) = \tfrac{1}{2} \quad \text{by MONO-REMOVE-BOTTOM}$$

and

$$P(D_2 \to c) \geqslant P(D_1 \to c) \geqslant \tfrac{1}{2} \quad \text{by LATER-NO-HELP,}$$

since $P(D_1 \to a) + P(D_1 \to c) = 1$ by MAJORITY. So $P(B_1 \to c) \geqslant \frac{1}{2}$ whenever δ and ε are sufficiently small. As in (ii), we can choose δ and ε so that $|\delta| + |\varepsilon| < 0.001$ and none of profiles B_1, B_1', B_3 and B_3' result in a tie, and we may suppose that $P(B_3 \to a) = 1$ and can deduce from the above that $P(B_1 \to c) = 1$. We can now follow the argument of (ii) to deduce that $P(B_6 \to a) = 1$, which implies that $P(D_3 \to a) = 1$ for suitable α, β, γ. Now

$$P(D_4 \to a) \geqslant P(D_3 \to a) = 1 \quad \text{by MONO-REMOVE-BOTTOM}$$

and

$$P(D_5 \to x) \leqslant P(D_4 \to x) = 0 \quad \text{by MONO-REMOVE-BOTTOM,}$$

where $x = b$ if $\mu \neq \alpha$ and $x = c$ if $\mu = \alpha$. But symmetry requires that $P(D_5 \to x) = \frac{1}{3}$, and this contradiction completes the proof of Theorem 3. □

We shall see by considering the Alternative Vote that MAJORITY, LATER-NO-HELP and LATER-NO-HARM are together compatible with all properties not specifically ruled out by Theorems 2 and 3.

5. Known non-Condorcet election rules

In the remainder of this paper we shall analyse a number of single-seat election rules, whose properties we summarize in Table 1. In this section we consider several known rules not satisfying Condorcet's principle. In most cases it

Table 1
Summary of properties of single-seat preferential rules

	FPP	PS	ApV	AV	ApAV	C–FPP C–PS	C–AV	B–F	Y	MM	K	PMM	MMG	QLTD	DAC
PLURALITY	√	√	√	√	√	√	√	√	×	×	×	√	√	√	√
MAJORITY	×	×	×	√	√	√	√	×	×	×	√	×	×	√	√
CONDORCET	×	×	×	×	×	√	√	√	√	√	√	√	■	×	×
SMITH–CONDORCET	×	×	×	×	×	√	√	×	×	×	√	×	×	×	×
EXCLUSIVE–CONDORCET	×	×	×	×	×	*	*	√	√	√	*	√	■	×	×
MONO-RAISE	√	√	√	×	√	√	×	√	√	√	√	√	√	√	√
MONO-ADD-TOP	√	√	√	√	×	⊗	⊗	⊗	√	√	×	⊗	√	×	√
MONO-REMOVE-BOTTOM	√	√	√	×	√	×	×	×	√	√	×	√	√	√	√
PARTICIPATION	√	√	√	×	×	⊗	⊗	⊗	⊗	⊗	⊗	⊗	⊗	×	√
MONO-RAISE-RANDOM	√	*	×	×	×	⊗	⊗	⊗	⊗	⊗	⊗	⊗	⊗	×	×
MONO-SUB-TOP	√	*	×	×	×	⊗	⊗	⊗	⊗	⊗	⊗	⊗	⊗	×	×
MONO-RAISE-DELETE	√	√	√	×	×	⊗	⊗	⊗	⊗	⊗	⊗	⊗	√	√	√
MONO-SUB-PLUMP	√	√	√	×	×	⊗	⊗	⊗	⊗	⊗	⊗	⊗	√	√	√
MONO-ADD-PLUMP	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√
MONO-APPEND	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√
LATER-NO-HELP	√	√	√	√	×	⊗	⊗	⊗	⊗	⊗	⊗	⊗	√	√	√
LATER-NO-HARM	√	×	×	√	×	⊗	⊗	⊗	⊗	⊗	⊗	⊗	×	×	×

Note: √ = Yes. ×, ⊗ = No (⊗ = No by Theorem 2). ■ = Yes if truncated preference listings are not allowed, otherwise no. * = Yes in special circumstances or with slight modifications, but no in general.
The double box delimits those properties that make sense even if truncated preferences listings are not allowed. The thick line separates global or absolute properties (above) from local or relative properties (below).

is fairly obvious which properties hold, but it is useful to have counterexamples for those that do not. Recall the definitions of C, v, $v(x)$ and $v_i(x)$ from Section 2.2.

Election 5		Election 6	
abc	25	*abc*	10
bac	30	*bca*	8
cab	45	*cab*	7

First-Preference Plurality (FPP), or First-Past-the-Post, elects the candidate x for whom $v_1(x)$ is largest. To see that this does not satisfy MAJORITY or CONDORCET, consider Election 5, where FPP chooses c, but MAJORITY requires that a or b should be elected, and a is the Condorcet winner. It is easy to see that FPP satisfies all the local properties we have mentioned, although it satisfies LATER-NO-HARM only if second and subsequent preferences are ignored totally, and not used to separate ties. (We shall describe election rules as if there is no tie; if there is a tie, that is, more than one candidate satisfies the specified criterion for election, then it is assumed that all such candidates are elected with equal probability.)

Point Scoring (PS) methods are those where, for some real numbers $a_1 > a_2 > \cdots > 0$, one elects the candidate x for whom $\sum_{i=1}^{|C|} a_i v_i(x)$ is largest. To see that such methods do not satisfy MAJORITY or CONDORCET, suppose $\frac{1}{2} + \varepsilon$ of the voters vote *abc* and $\frac{1}{2} - \varepsilon$ vote *bca*, where $\varepsilon > 0$. Then both MAJORITY and CONDORCET require that a should be elected, but any PS method will choose b provided that ε is small enough. The remaining properties are fairly obvious. Note that MONO-RAISE-RANDOM and MONO SUB-TOP do not hold in general, but the former holds if $a_i \geqslant 2a_i + 1$ for each i, and the latter holds if $a_1 \geqslant 2a_2$.

Approval Voting (ApV) [3] elects the candidate x for whom $v(x)$ is largest. (It was devised as a non-preferential system, and our use of it here is silly if truncated preference listings are not allowed, when $v(x) = v$ for all x.) Note that just as FPP is the limiting case of PS as $a_i \to 0$ for each $i \geqslant 2$, so ApV is the limiting case as $a_i \to 1$ for each i. ApV therefore has the same properties as PS. (For the failure of MAJORITY and CONDORCET, remove the last candidate from every ballot in the profile cited for PS.)

The Alternative Vote (AV) is the system where one repeatedly excludes the candidate with the smallest number of votes until there is only one candidate left, each vote being given at each stage to the first non-excluded candidate on the ballot. (If there is more than one candidate with the smallest number of votes, then one of them is chosen at random for exclusion. In practice one can stop as soon as some candidate has more than half the votes.) AV does not satisfy CONDORCET since it chooses b in Election 5, where a is the Condorcet winner. It is easy to see that it satisfies the other properties ticked in Table 1, although it satisfies LATER-NO-HARM only if ties are separated at random and not by looking ahead to later preferences. To see that it does not satisfy any other local properties, use Theorem 3. Alternatively note that, in Election 6 above, c is excluded and a is elected; but if two of the bca ballots are removed, or replaced by a or by abc or by anything else starting with a, then b is excluded and c is elected instead of a.

Approval AV (ApAV), which I proposed in [19], is a variant of AV in which one excludes at each stage the candidate x with smallest $v(x)$, stopping as soon as some candidate becomes the top non-excluded candidate on more than half the non-empty ballots. ApAV does not satisfy CONDORCET or MONO-ADD-TOP (add a single bc ballot in Election 5 to ensure that a, the Condorcet winner, is excluded first, so that b is elected; and then add a further two ba ballots to ensure that c is excluded first and a is elected). But, like AV, ApAV clearly satisfies MONO-APPEND and MONO-ADD-PLUMP, and unlike AV it also clearly satisfies MONO-RAISE and MONO-REMOVE-BOTTOM. It is easy to find examples to show that none of the other properties hold.

It is clear from Table 1 that the sets of properties satisfied by PS, ApV and ApAV are properly contained in those satisfied by FPP, FPP and QLTD (see Section 7) respectively. However, those satisfied by FPP and AV are maximal:

Theorem 4. *Among the properties listed in Table* 1, *there are precisely two maximal sets of mutually compatible properties that include both* LATER-NO-HELP *and* LATER-NO-HARM; *they are the sets satisfied by FPP and AV.*

Proof. Among sets of mutually compatible properties that include both LATER-NO-HELP and LATER-NO-HARM, we see from Table 1 and Theorems 2 and 3 that the set of properties satisfied by FPP is the unique maximal set not containing MAJORITY, and the set satisfied by AV is the unique maximal set containing MAJORITY. □

6. Condorcet-based election rules

6.1. Naïve rules

There is an obvious naïve method of modifying the above rules so that they satisfy Condorcet's principle: exclude all candidates not in the Condorcet top tier (closing up the gaps in the preference listings when candidates are excluded from them), and apply FPP, PS or AV to the remaining candidates. (One could use ApV or ApAV instead, but only if truncated preference listings are allowed.) The resulting rules are described in Table 1 as C–FPP, C–PS and C–AV. It is easy to see that they all satisfy PLURALITY. They do not satisfy EXCLUSIVE-CONDORCET: in Election 7, the Condorcet top tier is $\{a, b, c\}$, and all three rules elect c if k is large enough, whereas b is the unique Condorcet non-loser. However, the rules could easily be modified so that they do satisfy EXCLUSIVE-CONDORCET, just by redefining the Condorcet top tier in the event that there are Condorcet non-losers. Fishburn [9] has modified a method of Black [2, p. 66] in just this way; the modified method, called B–F in Table 1, is to declare the result a tie between all Condorcet non-losers if there are any, and otherwise to use PS (specifically, Borda counts).

Election 7		Election 8	
abc	$k+2$	*abc*	$k+2$
bca	$k+1$	*bca*	$k+1$
ca	k	*ca*	k
cba	$k+1$	*cba*	k

To see that none of these four Condorcet methods satisfy MONO-REMOVE-BOTTOM, note that in Election 8 the Condorcet top tier is $\{a,b,c\}$ and all four methods elect c if k is large enough. But if two *abc* ballots are removed, then b becomes the Condorcet winner. The same example shows that C–AV does not satisfy MONO-RAISE, since replacing two *abc* ballots by *cab* will cause a to be excluded instead of b, so that b is elected instead of c. It is easy to see that the other three methods satisfy MONO-RAISE, and all four satisfy MONO-ADD-PLUMP and MONO-APPEND. The failure of the remaining properties follows from Theorem 2, except for the failure of B–F to satisfy MAJORITY (and, therefore, SMITH–CONDORCET). This can be seen from Election 9 and modifications of it, where MAJORITY requires that a, b or c should be elected, but B–F will elect d.

Election 9

abcdef	102	*defabc*	98
cabdef	101	*defcab*	99
bcadef	100	*defbca*	100

6.2. *Other known Condorcet rules*

In [9], Fishburn considers nine election rules satisfying Condorcet's principle. Three of them fail the axiom of discrimination and so are not proper election rules according to our definition. Two others should be mentioned briefly. Roughly speaking, Nanson's rule [15, 16] is successively to delete the candidate with the smallest Borda count (recalculating the Borda counts after each deletion) until only one candidate is left; and Dodgson's rule [6] is to elect the candidate who can be made into a Condorcet non-loser by the smallest (fractional) number of transpositions of adjacent preferences in the preference listings in the profile. Fishburn gives examples [9, p. 478] to show that neither of these rules satisfies MONO-RAISE, and I shall not consider them further.

Of the four remaining rules discussed by Fishburn [9], we have already considered B–F, and we now consider the other three. Young's rule (Y) [21] elects the candidate who can be made into a Condorcet winner by deleting the smallest (fractional) number of ballots. The minimax rule (MM), ascribed in [9] to Condorcet [5], elects the candidate who can be made into a Condorcet winner by adding the smallest number of additional ballots. Kemeny's rule (K) [11, 12] first finds the linear ordering of the candidates that gives the largest agreement with the profile, and then elects the first candidate in that order; here the *agreement* of an order O with a profile P is $\sum_{\beta \in P} A(O,\beta)$, where $A(O,\beta)$ is the number of ordered pairs (x,y) of candidates such that x is above y in the order O and x is preferred to y by the ballot β.

Although Y and MM look similar, MM depends only on the net scores $n(x,y)$ — in fact, it elects the candidate x for whom maxna(x) (defined in Section 2.2) is minimal, whence the name *minimax* — whereas Y depends on the ballots themselves, since one cannot remove ballots that are not there. Nevertheless, the two methods have very similar properties. They both choose d in Election 9, showing that they do not satisfy MAJORITY or (therefore) SMITH–CONDORCET, whereas it is not difficult to see that K does satisfy SMITH–CONDORCET and hence MAJORITY. However, Y and MM satisfy EXCLUSIVE-CONDORCET, whereas K does not as it stands (see [9]), although it suffices to specify a suitable tie-breaking rule in order for K to do so. But none of the three satisfy PLURALITY, since all elect a in Election 4. The failure of K to satisfy MONO-ADD-TOP and MONO-REMOVE-BOTTOM follows from an example of Fishburn [9, p. 484], and the other entries in Table 1 all follow from Theorem 2.

6.3. *PMM, or plurality-minimax*

This is a rather heavy-handed modification of MM that satisfies CONDORCET, PLURALITY and MONO-REMOVE-BOTTOM (and is designed solely to prove that these three properties are mutually compatible). Say that a candidate x is *debarred by PLURALITY* if $v(x) < v_1(y)$ for some other candidate y. Note that a candidate who is debarred by plurality remains so if some complete preference listings are removed from the profile; that is, adding complete preference listings cannot cause a candidate to become debarred who was not so before. Let D_P denote the set of candidates x such that x is debarred by plurality in the profile obtained by removing all ballots in which x is bottom, below all other candidates. Clearly $D_P \neq C$, since any candidate with the largest number of first-preference votes after all complete preference listings have been removed from the profile is in $C \backslash D_P$. (If there are no truncated preference listings then $D_P = \emptyset$.) PMM elects the candidate x in $C \backslash D_P$ for whom minns(x) is maximal.

PMM clearly satisfies CONDORCET and PLURALITY. (A Condorcet winner cannot be in D_P.) It is easy to see that the winning candidate x cannot be moved into D_P if some ballots that have x bottom are removed, or if x is raised on some ballots, or if x is appended to some ballots that did not contain x, or if extra ballots are added that plump for x; and since these operations cannot decrease minns(x) nor increase minns(y) for any $y \neq x$, it follows that PMM satisfies the only four monotonicity properties that are possible for a rule that satisfies CONDORCET and PLURALITY.

6.4. *MMG, or maximin gross*

As we have seen, Condorcet-minimax (MM) elects the candidate x for whom maxna(x) is minimal, or, equivalently, for whom minns(x) is maximal. Like all Condorcet-based rules, it fails many forms of monotonicity, and so arguably it is not suitable for use when there are truncated preference listings. MMG is a workable system that, although it does not satisfy CONDORCET, can nevertheless be thought of as extending Condorcet's principle monotonically to profiles that include truncated preference listings. It elects the candidate x for whom mings(x) is maximal. Note that MMG agrees with MM, and hence satisfies CONDORCET and EXCLUSIVE-CONDORCET, if all preference listings are complete; therefore three properties fail by Theorem 2(a). Note also that adding a candidate y at the end of a ballot already containing x can raise mings(y) but cannot change mings(x); thus MMG satisfies LATER-NO-HELP but not LATER-NO-HARM, and it is easy to see by the same reasoning that it satisfies the other monotonicity properties ticked in Table 1. It also satisfies PLURALITY, because if $v(x) < v_1(y)$ then

$$\text{mings}(x) \leqslant v(x) < v_1(y) \leqslant \text{mings}(y).$$

6.5. *Unanswered questions*

Among many unanswered questions are the following:

Question 1. *Does there exist any election rule that satisfies* MAJORITY, CONDORCET, *and either or both of* MONO-ADD-TOP *and* MONO-REMOVE-BOTTOM?

Question 2. *Does there exist any election rule that satisfies all the properties satisfied by MMG, together with* MAJORITY?

Theorem 5. *If the answer to Question* 1 *is negative, then*

(i) *among the properties listed in Table* 1, *there are precisely three maximal sets of mutually compatible properties that include* CONDORCET; *they are the sets satisfied by MM, PMM and C–PS (modified so as to satisfy* EXCLUSIVE-CONDORCET);

(ii) *the set of properties satisfied by MMG is also a maximal set.*

Proof. Assuming that the answer to Question 1 is negative, and considering only sets of mutually compatible properties that include CONDORCET, Theorem 2 shows that MM satisfies the unique maximal set containing MONO-ADD-TOP; PMM, the unique maximal set containing MONO-REMOVE-BOTTOM but not MONO-ADD-TOP; and C–PS, the unique maximal set containing neither of these. Finally, a negative answer to Question 1 implies a negative answer to Question 2, and now Theorem 2(c) shows that MMG satisfies a maximal set of compatible properties. □

Needless to say, an affirmative answer to Question 1 would have no such consequences, and would leave open the determination of the maximal sets of mutually compatible properties satisfying CONDORCET.

7. New rules satisfying MAJORITY

7.1. QLTD, or quota-limited trickle-down

This rule is largely superseded by DAC (below), but I have included it here because it is simpler. For $x \in C$ and $t > 0$, let $v(x,t) = \sum_{i=1}^{\lfloor t \rfloor} v_i(x) + (t - \lfloor t \rfloor) v_{\lceil t \rceil}(x)$. Let $E(t) = \{x \in C : v(x,t) \geqslant \frac{1}{2}v\}$. If (exceptionally) $E(t) = \emptyset$ for all t, then QLTD elects the candidate x for whom $v(x) = v(x, |C|)$ is largest (as in ApV); otherwise let $t_0 = \inf\{t : E(t) \neq \emptyset\}$ and declare the result a tie between all candidates in $E(t_0)$. (Normally there will be just one.)

It is easy to see that QLTD satisfies PLURALITY, because if $v(x) < v_1(y)$ then $v(x,t) \leqslant v(x) < v_1(y) \leqslant v(y,t)$ for all $t \geqslant 1$, and $v(x,t) = tv_1(x) < tv_1(y) = v(y,t)$ if $t < 1$. It satisfies MAJORITY as well, because if $|S(X)| > \frac{1}{2}v$ then $v(x, |X|) > \frac{1}{2}v$ if $x \in X$ and $v(x, |X|) < \frac{1}{2}v$ if $x \notin X$, so that $t_0 < |X|$ and $E(t_0) \subseteq X$. Many forms of monotonicity are obvious, since if x is the candidate elected then the changes involved will not decrease $v(x,t)$ nor increase $v(y,t)$ for any $y \neq x$ or any t. But MONO-ADD-TOP fails in Election 9, where $t_0 = 2.97$ and a is elected, whereas if six extra *ad* ballots are added then $t_0 = 2.0$ and d is elected. LATER-NO-HARM and the other monotonicity properties that fail, do so for broadly similar reasons. QLTD does not satisfy CONDORCET even when there are no truncated preference listings, since it elects a in Election 10, in which c is the Condorcet winner.

Election 10		Largest acquiescing coalitions	
acbd	6	$\{a,b,c,d\}$	25
adbc	3	$\{a,b,c\}$	14
adcb	3	$\{a\}$	12
bcad	4	$\{a,c\}$	10
cabd	4	$\{a,d\}$	6
dbca	5		

7.2. DAC, or descending acquiescing coalitions

If $X \subseteq C$ then $A(X)$, the *coalition acquiescing to* X, comprises all voters who do not prefer any candidate not in X to any candidate in X. This includes $S(X)$ (as in Section 2.2) together with those voters who vote for fewer than $|X|$ candidates in total, these forming a proper subset of X. In DAC one first lists all the acquiescing coalitions in decreasing order of size, and then takes the intersection of the corresponding sets of candidates from the top down (ignoring any set that would give empty intersection) until one is left with a single candidate. For example, in Election 11, the eight largest acquiescing coalitions are as listed. The intersection of the largest two is $\{a,b,c\}$. The third, $\{d\}$, is disjoint from this, and so one ignores it. (It does not help in distinguishing between a, b and c.) Taking the intersection with the fourth gives $\{a,b,c\} \cap \{a,d\} = \{a\}$, and so a is elected. If the four *dabc* ballots were removed, then the result would be a tie between c (the result if there were $8+\varepsilon$ *cabd* ballots) and b (the result if there were $8+\varepsilon$ *dbca* ballots).

Election 11		Largest acquiescing coalitions			
adcb	5	$\{a,b,c,d\}$	30	$\{a,c\}$	8
bcad	5	$\{a,b,c\}$	13	$\{b,c,d\}$	8
cabd	8	$\{d\}$	12	$\{b,d\}$	8
dabc	4	$\{a,d\}$	9	$\{c\}$	8
dbca	8				

DAC elects a in Election 10, where c is the Condorcet winner, showing that it does not satisfy CONDORCET even when there are no truncated preference listings. If two of the *dabc* ballots in Election 11 were replaced by *acbd* then c would be elected, showing that DAC does not satisfy MONO-RAISE-RANDOM or MONO-SUB-TOP. The failure of LATER-NO-HARM is shown by the profile of Election 4 with two c ballots removed, where b is elected, but changing the 11 *ab* ballots to a causes a to be elected.

If $v(x) < v_1(y)$ and $y \notin X$ then

$$|A(X)| \leqslant v - v_1(y) < v - v(x) \leqslant |A(C\backslash\{x\})|,$$

and so if $|A(X)| \geqslant |A(C\backslash\{x\})|$ then $y \in X$. It follows from this that x is excluded before y, and so DAC satisfies PLURALITY. To see that it satisfies MAJORITY, note that if $|S(X)| > \frac{1}{2}v$ and $|A(Y)| \geqslant |A(X)|$ $(> \frac{1}{2}v)$, then either $X \subseteq Y$ or $Y \subset X$, and so the winning candidate must be in X. The fact that DAC satisfies LATER-NO-HELP and most monotonicity properties follows from the observation that an increase in $|A(X)|$ (keeping $|A(Y)|$ fixed for all $Y \neq X$) cannot decrease $P_E(x)$ for any $x \in X$, nor increase $P_E(x)$ for any $x \notin X$. To see that it satisfies PARTICIPATION, note that the effect of adding an extra ballot that is solidly committed to Y is to increase $|A(X)|$ for some sets X such that $X \subseteq Y$ or $Y \subset X$; the latter cannot decrease $P_E(x)$ for any x in Y, and the former cannot increase $P_E(x)$ for any x not in Y, and so together they cannot decrease $P(|E(Y)| \geqslant 1)$.

DAC seems to me to be a workable system (provided that the votes can be processed by computer) with almost all the properties that one could reasonably expect in a single-seat election rule. Among many unanswered questions, the following seem to me to be the most important:

Question 3. *Does there exist any single-seat preferential election rule that satisfies all the properties satisfied by DAC together with either* MONO-RAISE-RANDOM *or* MONO-SUB-TOP?

Question 4. Does there exist any multi-seat preferential election rule that satisfies DPC and all the properties satisfied by DAC except for MONO-REMOVE-BOTTOM, and that reduces to DAC in single-seat elections?

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