A MECHANISM FOR SELECTING PUBLIC GOODS WHEN PREFERENCES MUST BE ELICITED

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ABSTRACT

Decentralized provision of public goods provides individuals incentives to be free riders, which will lead to undersupply. If provision is centralized, individuals' preferences are not known to the authority or mechanism choosing the public goods bundle; hence efficient decisions cannot be guaranteed. The authority can ask people to report their preferences, but strategic rather than truthful responses must be expected.

Is it possible to construct a procedure which simultaneously induces participants to report their preferences correctly and uses such reported preferences to select a Pareto-optimal bundle of public goods? We address this question in a model where the public goods are financed through an existing tax system. That is, an individual's taxes depend only on the chosen bundle of public goods, not on anybody's expressed preferences.

We have succeeded in constructing such a procedure. It employs a form of weighted voting. Each individual has an exogenously given endowment of "influence points." A tentative decision is announced, and the individual then allocates these points among the various public goods and uses them to "vote" for an increase or decrease in the supply of each good. Influence points do not purchase votes for movement on a linear basis. As an individual spends more points on one good, the marginal value of an additional point decreases. Specifically, votes for movement equal the square root of the number of points expended. This decreasing productivity of influence point expenditures induces participants to spread out their allocations in a way which truthfully reveals their marginal valuations of the different public goods, which are the relevant aspects of their preferences.

If the votes cast in favor of increasing the supply of each public good exactly balance the ones cast in the opposite direction, an equilibrium is reached. This is the outcome of the procedure. It represents a Pareto-optimal decision.

Practical computation of such an equilibrium will involve all the problems associated with computing competitive equilibria in private goods markets. We present an algorithm which seems to work satisfactorily in a fairly general class of cases.

The procedure can be adapted to different sets of distributional objectives by varying the endowments of influence points assigned to individuals.

1. Introduction

Public goods choices challenge efficient decision making. Decentralized provision on a voluntary basis leads to undersupply, for each participant will seek to be a free rider. Consequently, most decisions to provide public goods are made on a collective or centralized basis, typically by a government. Centralized provision encounters a different set of difficulties: Individuals' preferences are not known to whatever authority or mechanism is making the choice, yet such information is required to guarantee an efficient outcome. In sum, when public goods decisions are decentralized, incentives are inappropriate; when they are centralized, information is insufficient. In this paper, we construct a method which deals with these two problems simultaneously and effectively. In the model considered, the revenue raising or tax system is external to the public goods decision, as is the case in most government jurisdictions.

For private goods the competitive market guarantees an efficient outcome if, as is traditionally assumed, each participant takes prices and other relevant factors about the external world as given. Individuals' decentralized, self-interested purchases and sales reveal all the information that is required to reach a Pareto-optimal outcome.

Public goods decisions are different. If the government, seeking to make an efficient decision, asks people to indicate their preferences for public goods, it will be clear that the responses will influence the outcome. There is nothing to prevent persons who recognize this effect from giving the responses which will best serve their interests, which may differ from their truthful responses. Even if the government then chooses an outcome which is efficient relative to the reported preferences, there can be no expectation of efficiency relative to the true preferences, which is what we want.

To illustrate the difficulty inherent in obtaining honest revelation of preferences, suppose that people are asked to report their marginal monetary valuation of a public good, with the understanding that the Lindahl solution will be applied.[1] Because there is an incentive to understate one's valuation, too little of the public good will be provided.

It should be noted that difficulties occur because preferences are unknown and, at the same time, efficiency is desired. If we care about only one of these factors, there is no problem. Given any particular set of individual preferences, the conditions for efficient provision of public goods are well known.[2] Similarly, as long as an optimal outcome is not required, it is possible to elicit honest preferences.[3] In particular, procedures based on majority vote will frequently guarantee that people react straightforwardly. But because such procedures give no information about strength of preference, efficient decisions cannot be assured.[4]

Can these two factors be dealt with simultaneously? Recent contributions to economic theory give some reason for optimism; procedures have been constructed which lead to Pareto-optimal outcomes and which, in a certain sense, elicit honest preferences.[5] An important feature of many of these procedures is that incentives to report preferences correctly are provided through a system of transfers of private goods or money. In other words, people pay taxes or receive subsidies in a way which depends on the preferences they express concerning public goods. Actual tax systems do not usually work this way.

We will assume that the tax system is given. This is not to say that the total amount of money spent on public goods need be fixed; this total can be a part of the decision. What is given is a rule which, for each potential public goods decision, specifies each individual's tax bill.

This implies that payment cannot be extracted from individuals directly on the basis of the preferences they express, only on the basis of the public goods bundle finally chosen. See Section 7 for further discussion of this condition.

Given this restriction, we cannot expect outcomes which are Pareto-optimal in the strict sense. The best we can achieve is an outcome which is Pareto-optimal subject to this constraint. Throughout this paper, we will use the phrase "Pareto optimal" in this constrained sense.[6]

Usually, there are many Pareto-optimal outcomes. The choice of one outcome from among these is a distributional issue. We want a procedure which can be used to accommodate varying distributional objectives.

The model we consider has a finite number of public goods, each of which is continuous in nature and can be provided in any non-negative amount. Some conditions are imposed on individual preferences, but these are not very restrictive.

To sum up, we make two crucial assumptions concerning the structure of the problem:

(i) Decisions are made by a central mechanism or authority which does not know people's preferences.

(ii) For every possible public goods decision, the financing mechanism and tax system are given.

We want to construct a procedure which meets two criteria:

(a) It produces a Pareto-optimal outcome when each participant reacts to the procedure in a self-interested manner.

(b) It can be adapted to different distributional objectives.

Underlying the analysis is an assumption that the government has to operate in the open; its procedures must be known to everybody ahead of time. In particular, the government does not have the option of first telling the participants that one procedure will be used, thereby inducing them to report their preferences correctly, and then processing this information by a different procedure so as to achieve optimality. We believe this assumption can be supported by ethical and legal, as well as practical, arguments.

Some of the language we use, such as "participants report their preferences" or "truthful reporting is elicited," may be somewhat metaphorical. The essential point is that individual responses to the procedure must convey sufficient information to enable the procedure to satisfy criterion (a). When we construct the procedure we must specify the format of the reporting. One possibility is to ask people to report their preferences directly. This involves an extensive amount of information, probably more than is needed and more than in practice can be transmitted. Instead, people can be required to reveal some aspect of their preferences, for example, their marginal valuations of the public goods at particular points. But the procedure can also decree that individual responses take some other form, if the ultimate purpose is better served that way. We will, however, continue to use the imprecise terminology of "reporting preferences."

We have succeeded in constructing a procedure satisfying (a) and (b). When the outcome is reached, individual responses will convey sufficient information and will constitute a (strict) Nash equilibrium. That is, any person who makes a unilateral change of response will lose. In this sense, the solution is consistent with the assumptions that preferences are unknown and that people act in a self-interested manner.[7]

The logical next question is: How do we actually compute the equilibrium? Our answer to this question, which is not an easy one, parallels that which obtains for competitive markets in private goods: Existence and optimality of equilibria can be proved, but the process by which an equilibrium can be reached is not well understood. We discuss this problem in Section 5, where we also present a simple iteration algorithm which appears to work in a fairly general case.

The paper is organized as follows: The model is presented in Section 2. In Section 3, we outline the procedure and give a heuristic proof that it works. The properties of the procedure are stated and proved formally in Section 4. Section 5 contains the discussion of computation of equilibria. The appropriate interpretation of the distributional objectives and the applicability of the model are considered in Sections 6 and 7. Section 8 presents some concluding remarks.

2. The Model

There are N individuals, denoted $1,\ldots,i,\ldots,N$. Decisions shall be made concerning K public goods, which are continuous in nature and can be provided in any non-negative amount. A typical decision is thus a vector $\underline{x} = (x_1,\ldots,x_K)$, where $x_k \geq 0$ for $k = 1,\ldots,K$. Nothing is said about the way the public goods are measured. (We shall later impose restrictions on people's valuation functions. These conditions may exclude otherwise feasible measurement scales.) For example, some goods can be measured in terms of money spent and others in physical units. We can let two or more of our "goods," that is, two or more dimensions of the K-dimensional space, represent different aspects of what is essentially the same public good. For example, one dimension may be the amount of money spent on the local public library, while one or more other dimensions represent aspects of the institution's purchasing policy (proportion of fiction versus non-fiction, modern versus classical literature, etc.). The only restriction is that the issues must be formulated in such a way that the outcome of necessity will be described by one and only one point $\underline{x} = (x_1,\ldots,x_K)$. That is, every possible system of decisions must correspond to exactly one such point.

No problem will result if there is an upper bound on the amount which can be provided of one of the goods. In fact, our later assumptions imply that the set of Pareto-optimal points is bounded. Hence we can impose such an upper bound without losing anything of interest. More generally, the set of feasible decisions can be any compact and convex subset of the K-dimensional Euclidean space whose interior is non-empty.[8]

The condition that the levels of public goods be non-negative is not very restrictive, because the scale of measurement is arbitrary. If it is possible to disinvest in a certain public good, "zero" on our scale shall not represent zero expenditure, but the largest possible amount of disinvestment. For example, the decision variable might be the amount of money to be spent this year on the acquisition of open space or park land for the city. Perhaps some people think not only that no money should be spent for this purpose, but that parks should be sold and the proceeds used for tax rebates. Then the appropriate zero point on the scale might be "sell all the parks." Our model cannot allow goods for which unlimited disinvestment is possible, but that is hardly a severe restriction.

Individual i is supposed to evaluate potential public goods decisions according to a valuation function v_i. We are concerned only with decision making under certainty, hence v_i is determined only up to strictly increasing transformations.[9] As mentioned in the Introduction, we assume that for every potential decision $\underline{x}$, it is decided in advance how the corresponding public goods will be financed and how much each person will have to pay in taxes. The function v_i is supposed to take account of the taxes person i pays for each $\underline{x}$. It also must capture any indirect effect of the decision $\underline{x}$ on the individual's welfare. For example, the decision may influence prices and market conditions for private goods.[10]

The model does not make specific assumptions about the nature of the tax system. Perhaps the simplest would be a system in which each person pays a predetermined share of the cost of the final decision. These shares must be positive and sum to one; in determining them, one can take into account people's initial wealth or whatever other factors are deemed relevant. Alternatively, we can have a tax system in which the cost of any one public good is shared in a given way, but the sharing rule may be different for different goods. In general, each person's tax is an arbitrary function of the chosen public goods bundle, subject only to the constraint that the sum of the taxes always must cover the cost of the decision. An individual's tax may even be negative for certain public goods bundles, in which case it represents a subsidy.

The assumptions we have made concerning the tax system do not rule out the possibility that certain aspects of the tax system are variables to be determined as a part of the public goods decision . For example, some measure of the degree of progressivity of the tax system could be included as a coordinate of $\underline{x}$. It will still be true that for any potential decision $\underline{x}$, each individual's tax is determined.[11]

Some of the public goods can represent income transfers to or from specific individuals or groups. Our model can also be applied to a situation where income transfer is the only issue. The decision $\underline{x}$ must then describe each individual's wealth after redistribution.[12] It is reasonable to assume that people are somewhat altruistic but mainly concerned with their own wealth, and this can easily be represented in our model.

It is possible that the taxes are independent of the decision. Then total resources will be fixed, and only decisions which satisfy the resource constraint are feasible.[13]

Our model can also be used to address situations in which no taxes are levied on the individuals who make the decisions. The issues can be purely non-monetary, or the financing can be unrelated to the participants' private economy. As an example of this we can mention decision making in university departments; here a fixed budget is allocated, and decisions without financial implications are also made (about curriculum, etc.).

We assume that each v_i is continuously differentiable and strictly concave. Moreover, we assume that everybody wants a little of every public good, but at some point the marginal net valuation of a good becomes negative, either because no more of this commodity is wanted, or because an additional unit is not worth the added tax burden or the sacrifice which must be made in terms of other public goods. Formally, this amounts to the following:

(1) For any k and any possible decision $\underline{x} = (x_1, \ldots, x_K)$ with $x_k = 0$,

$$\left.\frac{\partial v_i}{\partial x_k}\right|_{\underline{x}} > 0.$$

(2) For any k there exists a positive number x_k^* such that, for any possible decision $\underline{x} = (x_1, \ldots, x_K)$ with $x_k \geq x_k^*$,

$$\left.\frac{\partial v_i}{\partial x_k}\right|_{\underline{x}} < 0.$$

The numbers $x_1^*, \ldots, x_K^*$ can be chosen independently of i. Conditions (1) and (2) are needed for some but not all of our results.[14]

The assumptions imply that v_i has a unique, interior maximum. This point is the ideal decision from i's point of view.

The possibility of finding concave valuation functions may depend on the way the public goods are measured. (Concavity is invariant under linear changes of scale, but there may exist natural alternative scales which are not linear transformations of each other.) If we start with a model with both private and public goods and derive preferences over public goods bundles from preferences over both types of goods, the existence of concave functions v_i can be proved under relatively standard assumptions. For example, this conclusion follows if public goods are produced at fixed cost, taxes are proportional, and individual preferences over bundles of private and public goods can be represented by concave utility functions.[15]

3. Outline of the Procedure

As noted in the Introduction, the difficulty in constructing a successful procedure arises because it must simultaneously satisfy two criteria: Individuals' honest preferences must be elicited, and in light of these preferences an efficient bundle of public goods must be selected. Below, we will consider these two criteria separately and discuss the restriction each one imposes on the procedure. Thereafter, we bring the two together and conclude that there is a way to satisfy both. Moreover, there is essentially only one way to solve the problem. But first we shall make some general remarks on the structure of the procedure.

The procedure takes the following form: A tentative decision $\underline{x} = (x_1,\ldots,x_K)$ is announced. Each individual reacts to this announcement by requesting an increase or decrease in the various public goods. We will use the symbol $\underline{b}_{i.} = (b_{i1},\ldots,b_{iK})$ to denote i's request; b_{ik} can be positive or negative and represents the change i asks for in the provision of good k. The number b_{ik} will be referred to as the number of "votes" person i uses to move the decision concerning good k.

In analogy with private goods markets, we could say that person i "purchases" a change in the provision of each public good. Clearly not all purchases can be permitted; some restrictions must be imposed, just as prices and budget restrict an individual's purchases of private goods. When constructing the procedure, we are free to choose the restrictions that shall be imposed on an individual's "purchase." These restrictions constitute the "rules of the game" and are the critical aspect of the procedure; hence they are the subject of most of the discussion below. At this point, we only note that the rules cannot in any way depend on individuals' preferences, since these are unknown to the authority that gives the rules.

The announced, tentative decision $\underline{x}$ and the votes of all the participants determine another possible decision $\underline{y} = (y_1,\ldots,y_K)$ by

$$(3) \quad y_k = x_k + \sum_{i=1}^{N} b_{ik}, \text{ for } k = 1,\ldots,K.$$

This $\underline{y}$ shall be thought of as the final decision. In general, $\underline{y}$ will be different from $\underline{x}$. But if it so happens that $\underline{y}$ equals $\underline{x}$, that is, if the changes requested by the participants precisely cancel each other, then we say that $\underline{y} = \underline{x}$ is an equilibrium. We declare it to be the outcome.[16] Hence the criterion for an equilibrium is the following:

$$(4) \quad \sum_{i=1}^{N} b_{ik} = 0, \text{ for } k = 1,\ldots,K.$$

We could describe our procedure formally by specifying the set from which the vector $\underline{b}_{i.}$ must be chosen. The specification of that set for each individual would fully describe the procedure. We shall take a somewhat different approach to facilitate intuitive understanding.

Our procedure endows each individual with a budget of total influence. This influence shall be "spent" on achieving changes in the public goods bundle. Individual i is endowed with A_i influence points. The numbers $A_1,\ldots,A_N$ are predetermined and non-negative, and at least one A_i is positive. These numbers represent people's relative claims on influence or power, based on ethical or other considerations external to our model. An important special case is the one of equal endowments. If $A_i = 0$, individual i has no influence and will be ignored by the procedure. When the tentative decision $\underline{x}$ is announced, individual i "spends" the A_i points by dividing them among the K public goods, indicating whether more or less is desired of each good.

Formally, i chooses a vector $\underline{a}_{i.} = (a_{i1},\ldots,a_{iK})$, where a_{ik} is the influence points i spends on good k. We adopt the convention that a positive (negative) value of a_{ik} indicates that i wants more (less) of good k. The "budget constraint" which must be satisfied by $\underline{a}_{i.}$ is

(5) $$\sum_{k=1}^{K} |a_{ik}| \leq A_i \quad .$$

The influence points are used to "purchase" votes or changes in the public goods bundle. In contrast to ordinary market situations, we will not restrict ourselves to linear price systems; indeed, we shall see later that we will have to rely on a non-linear system. We assume that the pricing relationship is given by an arbitrary function f, which determines how the chosen numbers a_{ik} translate into the votes b_{ik}. That is,

(6) $$b_{ik} = f(a_{ik}) \ .$$

The sign convention on a_{ik} implies that f(a) always must have the same sign as a. Strictly speaking, the function f is the inverse of a price system; f(a) is the amount one can buy for an expenditure of a. The price system and f are two ways of representing the same reality, and we shall use them interchangeably.[17]

Once the function f is given, the description of the procedure is complete. The individuals choose "expenditures," that is, the $\underline{a}_{i.}$ vectors. These are translated into "purchases" or votes $\underline{b}_{i.}$. The decision $\underline{y}$ is given by (3), and when $\underline{y} = \underline{x}$ an equilibrium and final outcome is reached.

Let us now explore the restrictions placed on the procedure (and therefore on f) by the requirements of efficiency and compatibility with individual self-interested behavior.

Pareto optimality. In our model, the criterion for Pareto optimality is particularly simple: A point is Pareto optimal if and only if it maximizes some weighted sum of individual valuations. To state this formally, let $\lambda_1,\ldots,\lambda_N$ be non-negative numbers, at least one of which is positive, and define a function v_Λ by

$$(7)\quad v_\Lambda(\underline{x}) = \sum_{i=1}^{N} \lambda_i v_i(\underline{x}) .$$

Then v_Λ is strictly concave and satisfies (1) and (2); hence it has a unique maximum point. This point is Pareto optimal. Conversely, it is not difficult to prove that for any Pareto-optimal point, there exists a function of the form (7) which is maximized at that point.[18]

Since v_Λ is differentiable and concave, the first-order conditions are necessary and sufficient for an interior maximum. Because of (1), we can rule out corner solutions. Therefore, $\underline{x}$ is a maximum of v_Λ if and only if

$$(8)\quad \frac{\partial v_\Lambda}{\partial x_k} = \sum_{i=1}^{N} \lambda_i \frac{\partial v_i}{\partial x_k} = 0, \text{ for } k = 1,\ldots,K,$$

where the derivatives are evaluated at $\underline{x}$. Intuitively, this says that the sum of the marginal valuations, when appropriately weighted, of those who want more and of those who want less of a particular good must cancel each other exactly. The condition is closely related to well-known characterizations of optimality in public goods provision when the division of costs is not exogenous.[19]

Pareto optimality of an equilibrium outcome is now guaranteed if and only if the two criteria (4) and (8) coincide. If all the numbers b_{ik} could depend on all the functions $v_1,\ldots,v_N$, there would be many ways to achieve this. But for any i, $b_{i1},\ldots,b_{iK}$ are determined by person i, who does not know the functions v_j for $j \neq i$ and whose actions cannot depend on these functions. Therefore, in order to guarantee that (4) and (8) coincide, we must require that they be equal term by term. That is, we must have

$$(9) \qquad f(a_{ik}) = b_{ik} = \lambda_i \frac{\partial v_i}{\partial x_k}, \quad \text{for all } i \text{ and } k,$$

where the derivatives are evaluated at the equilibrium outcome $\underline{x} = \underline{y}$.[20]

It is obvious that in order to guarantee Pareto-optimal outcomes, a procedure must have access to information about individuals' relative marginal valuations of the various goods.[21] That is, answers must be known to questions of the following type: Given the financing scheme, how much would you be willing to give up of public good 1 in order to get a unit more of public good 2? The argument above shows that this information must take a special form: The votes $b_{i1},\ldots,b_{iK}$, which represent person i's desired changes in the amounts of public goods provided, must be proportional to i's marginal valuations. In this condition lies the first restriction on the procedure and on f.

Individual self-interested behavior. To deduce the second restriction on f, we look at the decision problem for the self-interested individual i. The outcome $\underline{y}$, as given by (3), depends on the announced tentative decision $\underline{x}$, the reaction of everybody but i, and i's own response. Individual i can in no way influence the first two factors and should therefore take them as given. From (3), it follows that the problem is to choose the vector $\underline{a}_{i.} = (a_{i1},\ldots,a_{iK})$ which will

(10) maximize $v_i(\underline{z}^{(i)} + \underline{b}_{i.})$,

subject to (5). Here $\underline{b}_{i.} = (b_{i1},\ldots,b_{iK})$ is given by (6) and $\underline{z}^{(i)} = (z_1^{(i)},\ldots,z_K^{(i)})$ is the sum of the effects i does not control. That is,

$$z_k^{(i)} = x_k + \sum_{j \neq i} b_{jk} .$$

Let us assume that f is differentiable and ignore the possibility of corner solutions.[22] Then a solution to the constrained maximization problem given by (5) and (10) must satisfy the first-order condition

$$(11) \quad \frac{\partial v_i}{\partial x_k} \cdot f'(a_{ik}) = \sigma_{ik}\mu_i, \text{ for } k = 1,\ldots,K.$$

Here σ_{ik} is the sign of a_{ik}, that is, $\sigma_{ik} = 1$ if $a_{ik} > 0$ and $\sigma_{ik} = -1$ if $a_{ik} < 0$. The derivative of v_i is evaluated at the final decision $\underline{y}$. The positive number μ_i represents the shadow price of the constraint (5). That is, μ_i is the marginal value, in terms of v_i, of an additional influence point. Equation (11) represents the familiar condition that at the optimum the marginal value of spending another influence point must be the same for each good, and this common value is equal to the shadow price.[23]

Thus we have found the restriction imposed on f by the assumption of individual self-interested behavior.

<u>Failure of a linear price system</u>. Returning to the question of what form should be employed for the function f, we might at first think of making it the identity function $f(a) = a$ for all a. This would imply that changes in the provision of public goods, as given by the votes b_{ik}, could be "purchased" at the constant price of 1 influence point per unit change.[24] In other words, individual i can influence the public goods vector $\underline{x}$ by the total of A_i units, these units being divided among the K goods in any way.

This approach does not work. It is easy to see that if i's influence is relatively small, and if the announced decision $\underline{x}$ is some distance away from i's optimal point, the optimal action will usually be to put all the A_i influence points on the single public good that yields the highest marginal valuation.[25] In violation of (9), the numbers b_{ik} will not be proportional to i's marginal valuation of the goods. Indeed, the only information that is revealed is which good has highest marginal valuation. The necessary information not being available, Pareto optimality cannot be guaranteed.

It should be noted that the problem outlined in the previous paragraph has a parallel in private goods markets: Assume that you are endowed with a bundle of private goods which is very different from your optimal bundle at the prevailing prices, and assume that you have a small amount of money to spend but cannot make other transactions. Then you will most likely spend all the money on one good, namely the one for which marginal utility per dollar is highest. With private goods, people are in practice not confronted with this problem. Each individual is permitted to purchase the bundle which is optimal at the prevailing prices. For public goods, however, the problem is real. Unless preferences are remarkably similar among individuals, at least some people will be far from their optimal points.

Straightforward per-unit pricing fails to elicit the required information about individuals' preferences. Instead we must find a pricing mechanism that will lead an individual to spread the influence points among the public goods. Moreover, they must be spread in such a way that (9) holds.

<u>The solution, the square-root formula</u>. To assure that influence points are spread around, we must offer diminishing marginal returns to the influence points spent on purchasing the changes given by b_{ik}. In other words, the

price of a unit change increases as more units are purchased. Indeed, if the first unit is sufficiently cheap, individual i will normally spend some influence points on every public good.[26] Formally, this means that f(a) should increase at a decreasing rate when a is positive and increasing, with a parallel property for negative a.

The question then is: How quickly should the "marginal productivity" of the influence points decrease? In fact, all requirements are satisfied if f is the square-root function. As we shall see later, this function is essentially the only one that works. Since the numbers a_{ik} can be both positive and negative, the formal definition of f is

$$f(a) = \begin{cases} \sqrt{a} & \text{for} \quad a \geq 0 \\ -\sqrt{-a} & \text{for} \quad a < 0 . \end{cases} \tag{12}$$

There are several ways to see that this function will indeed solve the problem. First we give an informal argument. Assume that there are only two public goods, and suppose that individual i's marginal valuation of each good is positive and constant over the relevant range.[27] We write

$$c_{ik} = \frac{\partial v_i}{\partial x_k} ,$$

for k = 1,2 and all i. The self-interested individual i then faces the problem:

$$\text{maximize } c_{i1}\sqrt{a_{i1}} + c_{i2}\sqrt{a_{i2}}$$

subject to

$$a_{i1} + a_{i2} \leq A_i .$$

It is clear that the constraint will be binding. We substitute for a_{i2} and get:

$$\text{maximize } c_{i1}\sqrt{a_{i1}} + c_{i2}\sqrt{A_i - a_{i1}} .$$

Setting the derivative equal to 0, gives

$$\frac{c_{i1}}{2\sqrt{a_{i1}}} - \frac{c_{i2}}{2\sqrt{A_i - a_{i1}}} = 0.$$

By rearranging terms, we get

$$\frac{\sqrt{a_{i1}}}{\sqrt{A_i - a_{i1}}} = \frac{c_{i1}}{c_{i2}} .$$

But the numerator and denominator on the left-hand side are b_{i1} and b_{i2}, respectively; that is, they are the changes induced in the two goods. These values are proportional to the marginal valuations, as is required.

Then we give a formal proof. The number of public goods is now arbitrary, and no special assumptions are made concerning marginal valuations. The derivative of f is easily seen to satisfy:

(13) $$f'(a) = \frac{\sigma}{2f(a)} ,$$

for $a \neq 0$, where $\sigma = 1$ for $a > 0$ and $\sigma = -1$ for $a < 0$. Except when the marginal valuation of a good is 0, i will not choose a corner solution. This is true because the derivative of f is infinite at 0; hence spending a few influence points on any good yields substantial returns. Therefore, individual self-interested behavior is described by (11). From (11) and (13) we get

$$\frac{\partial v_i}{\partial x_k} = 2\mu_i f(a) .$$

Thus (9) holds, with $2\mu_i = 1/\lambda_i$. This proves that the function f, as given by (12), satisfies the requirements.[28]

Uniqueness of the solution. Essentially, the square-root function given by (12) is the only function f which guarantees Pareto optimality in the framework of our model.[29]

We know that equations (9) and (11) both must hold. Together they imply, for all i and k,

(14) $f(a_{ik}) \cdot f'(a_{ik}) = \sigma_{ik}\lambda_i\mu_i$.

For a given i, the right-hand side of (14) depends on k only through σ_{ik}, that is, the sign of a_{ik}. If,for a moment, we restrict ourselves to positive values of a_{ik}, the right-hand side is independent of k. Since we do not know v_i, we must be prepared to encounter any vector $\underline{a}_{i.} = (a_{i1},\ldots,a_{iK})$ which satisfies (5). In order to guarantee that (14) holds, we must therefore construct the function f such that the left-hand side does not depend on a_{ik}.[30] Hence there must exist a positive constant C such that

(15) $f(a) \cdot f'(a) = C$, for all $a > 0$.

The left-hand side of (15) is half the derivative of the function $(f(a))^2$. Hence this function has a constant derivative and must be linear. Since $f(a) > 0$ for $a > 0$, we conclude that f must satisfy

(16) $f(a) = \sqrt{C_1 a + C_2}$, for $a > 0$.

Here $C_1 = 2C$ is a positive constant, and C_2 is a constant. If $C_2 < 0$, f(a) is not defined for small $a > 0$. If $C_2 > 0$, (14) is satisfied. But if we go back to the original requirements, we see that the maximization problem given by (5) and (10) will sometimes have a corner solution, and (9) will not hold. Hence we must have $C_2 = 0$.

An analogous argument can be applied to (14) for the case $a_{ik} < 0$, and we can deduce a formula similar to (16). Again, C_2 must be 0, and C_1 must be the same in the two cases in order for (14) to hold. Thus we have proved that the function f must be the one given by (12), except for the constant C_1.

The numbers $A_1,\ldots,A_N$ are chosen when the procedure is designed. (This choice is exogenous to the aspects of the model discussed in this section.) It is easy to see that a change in the constant C_1 always can be compensated by a change in $A_1,\ldots,A_N$; that is, the two changes taken together do not alter the procedure. This proves that there is no loss of generality in setting $C_1 = 1$; everything we might want to do by varying C_1 can be done by varying $A_1,\ldots,A_N$ instead.[31]

This completes the argument that the function f given by (12) essentially is the only solution to the problem.[32]

An example. In order to illustrate the equilibrium concept, we give a simple example. We assume that there are two public goods and three individuals; this is the simplest non-trivial case. Preferences are of the following form: Each individual has an optimal point; individual i's optimal point is denoted $\underline{x}^{(i)}$. Preferences depend only on the distance to the optimal point, that is, i prefers one point to another if the former is closer to $\underline{x}^{(i)}$. This uniquely specifies the preferences; we can define $v_i(\underline{x}) = -\|\underline{x} - \underline{x}^{(i)}\|^2$. The indifference curves will be concentric circles.[33]

For any tentative decision $\underline{x}$, $\underline{a}_{i}$. will be chosen such that $\underline{b}_{i}$. points in the direction from $\underline{x}$ to $\underline{x}^{(i)}$.

Now assume $\underline{x}^{(1)} = (1,1)$, $\underline{x}^{(2)} = (1,10)$ and $\underline{x}^{(3)} = (10,1)$, and let $A_i = 1$ for $i = 1,2,3$. The average of the three optimal points is (4,4); therefore, it seems natural to try that point in order to see if it is an equilibrium. This is illustrated in Figure 1 and Table 1. It turns out that the sample point (4,4) is not an equilibrium; the sum of the votes $\underline{b}_i$. points from (4,4) towards $\underline{x}^{(1)}$ and the origin. The equilibrium is given by $x_1 = x_2 = \frac{11}{2} - \frac{3}{2}\sqrt{3} = 2.9019$; see Figure 2 and Table 2.

Insert Figure 1 and Table 1 About Here

Insert Figure 2 and Table 2 About Here

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4. Properties of the Procedure

We describe the procedure formally in this section and derive a number of its properties. The presentation at points parallels that of Section 3; in order to ease understanding, there is some repetition.

Our first result states that if $\underline{x}$ is an equilibrium of the procedure, as described in Section 3 and defined formally below, then $\underline{x}$ and the corresponding individual responses constitute a Nash equilibrium of an appropriately defined game, and hence no person can gain by a unilateral change of response. Second, we prove that an equilibrium will always be a Pareto-optimal decision. Conversely, our third proposition states that for any Pareto-optimal point there exists a distribution of influence among the individuals such that the given point becomes an equilibrium of the procedure. The last two statements correspond to the first and second fundamental theorems of welfare economics. The last result is non-vacuous, since Pareto-optimal points clearly exist and can be obtained by maximizing functions of the form (7). But in order to guarantee that the first two results are true in more than a trivial sense, we must prove that equilibria always exist. This is our fourth proposition.

The numbers $A_1,\ldots,A_N$ are fixed and are supposed to be given by considerations outside the model. They are non-negative, and at least one of them is positive. For each i, we define $B_i = \sqrt{A_i}$. The distribution of influence can be described either by the numbers A_i or by B_i. We will use either, as convenience dictates. The function f and the numbers a_{ik} were used in Section 3 to facilitate intuitive understanding; they play no role in the formal definitions in this section.

A tentative decision $\underline{x}$ is announced. For each i, individual i chooses a vector $\underline{b}_{i.} = (b_{i1},\ldots,b_{iK})$, subject to

$$(17) \quad \sum_{k=1}^{K} b_{ik}^2 \leq B_i^2 = A_i .$$

This means that the length of the vector $\underline{b}_{i.}$ must not exceed B_i. The vector is chosen so as to maximize

$$(18) \quad v_i(\underline{y}) ,$$

where $\underline{y} = (y_1,\ldots,y_K)$ is given by

$$(19) \quad y_k = x_k + \sum_{i=1}^{N} b_{ik}, \text{ for } k = 1,\ldots,K.$$

Here the numbers b_{jk} for $j \neq i$ are supposed to be determined by similar maximizations. It is easy to see that this parallels the description in Section 3; given (6) and (12), (17) is equivalent to (5).

When characterizing the solution of this maximization problem, we must distinguish between two cases. We focus on one individual i, and assume that $\underline{x}$ and $\underline{b}_{j.}$ for $j \neq i$ are given. If it is possible, without violating (17), to choose $\underline{b}_{i.}$ such that $\underline{y}$ becomes equal to the maximum of v_i, then i will obviously choose this $\underline{b}_{i.}$. In this case, (17) need not be a binding constraint. Otherwise, that is, if it is not possible to reach the maximum of v_i, individual i could always benefit from increasing the absolute value of some b_{ik}. Hence (17) will be binding. Since $v_i(\underline{y})$ is a concave function of $\underline{b}_{i.}$, the first-order conditions are necessary and sufficient for a solution, and it is easy to prove that these conditions can be written:[34]

$$(20)\quad b_{ik} = \lambda_i \frac{\partial v_i}{\partial x_k}, \text{ for } k = 1, \ldots, K.$$

Here λ_i is a positive constant chosen so that (17) holds with equality. The derivatives are evaluated at $\underline{y}$. This equation corresponds to (11); the equivalence is proved by using (12) and setting $\lambda_i = 1/(2\mu_i)$.

The basic characteristics of the individuals are their preferences; the functions $v_1, \ldots, v_N$ are just representations of these preferences. We do not want the results of our model to depend on the particular representations chosen. This requirement is satisfied. The solution to the maximization problem given by (17) - (19) does not change if v_i is replaced by a differentiable, strictly increasing transformation of v_i. In the case where i's optimal point can be reached, this is obvious, since v_i and the transformation are maximized at the same point. When (20) applies, the transformation has the effect of multiplying $\partial v_i/\partial x_k$, for k = 1, ...,K, by a positive constant. This is compensated by a change in λ_i, and $\underline{b}_{i.}$ remains unchanged.

<u>Definition</u>

A feasible decision $\underline{x}$ is an <u>equilibrium</u> of the procedure if there exist numbers b^*_{ik}, for i = 1,...,N and k = 1,...,K, such that:

(i) For each i, the maximization problem given by (17) - (19) has a solution at $\underline{b}_{i.} = \underline{b}^*_{i.}$, when all $\underline{b}_{j.}$ for $j \neq i$ are fixed with $\underline{b}_{j.} = \underline{b}^*_{j.}$.

(ii) $\underline{y} = \underline{x}$, where $\underline{y}$ is given by (19) with b^*_{ik} substituted for b_{ik}.

If $\underline{x}$ is an equilibrium and is no individual's optimal point, then all the numbers b^*_{ik} will be given by (20). Hence i's equilibrium response $\underline{b}_{i.}$ depends only on $\underline{x}$, A_i and v_i. That is, it depends only on common information and on i's own characteristics. If, on the other hand, $\underline{x}$ is i's optimal point, i's response $\underline{b}^*_{i.}$ is chosen so as to "balance" the responses and make $\underline{y}$ equal to $\underline{x}$; hence $\underline{b}^*_{i.}$ will depend on all the data of the problem.

Note that the equilibrium concept is invariant to changes of scale in the numbers $A_1,\ldots,A_N$ (or $B_1,\ldots,B_N$). To state this formally, assume that $\underline{x}$ is an equilibrium, let α be a positive number, and let $\beta = \sqrt{\alpha}$. Suppose that A_i is replaced by αA_i for $i = 1,\ldots,N$ (or, equivalently, that B_i is replaced by βB_i). Then $\underline{x}$ is still an equilibrium. In the definition of equilibria, b^*_{ik} must be replaced by βb^*_{ik} for all i and k. The statement is easily proved; the two cases referred to in the discussion of the maximization problem (17) - (19) must be considered separately. Given this result, it follows that the numbers A_i only represent people's relative power or influence; the absolute size of these numbers has no significance.

Now we are in the position to state precisely the sense in which an outcome of our procedure is consistent with individual, self-interested behavior.

<u>Proposition 1</u>

Let $\underline{x}$ be an equilibrium in the above sense, and let b^*_{ik}, for $i = 1,\ldots,N$ and $k = 1, \ldots,K$, be the corresponding numbers, as given by the definition.

Consider the following game: Each person i submits a vector $\underline{b}_{i.}$ which satisfies (17). Then $\underline{y}$ is computed by (19), and the decision $\underline{y}$ is implemented.

In this game, the responses given by $\underline{b}_{i.} = \underline{b}^*_{i.}$ for all i constitute a (strict) Nash equilibrium. That is, any individual who makes a unilateral change of response is made worse off.

The proof is trivial, since the proposition is only a restatement of the definition of an equilibrium. The converse statement is equally obvious: If there is given a feasible decision $\underline{x}$ and individual responses b_{ik} which make $\underline{y}$ equal to $\underline{x}$ and constitute a Nash equilibrium of the game described in Proposition 1, then $\underline{x}$ is an equilibrium in the sense of our definition.

In the Nash equilibrium described in Proposition 1, everybody is supposed to act honestly, that is, all the numbers b^*_{ik} are found by solving (17) - (19). This is not an essential assumption. Neither person i nor the procedure knows the functions v_j for $j \neq i$. Hence i can always assume that v_j is such that the given response $\underline{b}^*_{j.}$ is an honest one. Therefore, correct revelation of preferences is i's optimal response in the game described in the proposition, regardless of the correctness of other individuals' reporting.

Even if $\underline{x}$ is an equilibrium, it is possible that two or more persons could all gain by making coordinated changes in their responses. Therefore, we have to assume non-coordinated (or competitive) behavior in order to conclude that an equilibrium outcome is consistent with individual self-interested behavior. (Competitive markets for private goods, by analogy, break down if cartels are formed.) But it is not necessary to assume that the number of participants is "large" in any sense; the proposition is true even for small N.[35]

Proposition 2

If $\underline{x}$ is an equilibrium, then $\underline{x}$ is Pareto optimal.

Proof

If $\underline{x}$ maximizes some v_i, any change away from $\underline{x}$ will be opposed by person i, and optimality follows.

Otherwise, the numbers b_{ik}^* of the definition are all given by (20). By (ii) of the definition,

$$\sum_{i=1}^{N} b_{ik}^* = 0$$

for all k. Then (20) implies

$$(21) \quad \sum_{i=1}^{N} \lambda_i \frac{\partial v_i}{\partial x_k} = 0, \text{ for } k = 1,\ldots,K.$$

The numbers λ_i are all non-negative, and if $A_i > 0$ then $\lambda_i > 0$. Hence $\lambda_i > 0$ for at least one i. Therefore, (21) is exactly the criterion for the point $\underline{y} = \underline{x}$ being Pareto optimal; see the discussion of equations (7) and (8) in Section 3. The proof is complete. ||

So far, the numbers $A_1,\ldots,A_N$ have been regarded as given and fixed. In the next result we do not make this assumption. Instead, the problem now is to find $A_1,\ldots,A_N$ such that a given Pareto-optimal point can be obtained as an equilibrium.

Proposition 3

Let $\underline{x}$ be a Pareto-optimal point. Then there exist non-negative numbers $A_1,\ldots,A_N$, at least one of which is strictly positive, such that $\underline{x}$ is an equilibrium of the corresponding procedure.

Proof

If $\underline{x}$ is the optimal point for some i, we let $A_i = 1$ and $A_j = 0$ for $j \neq i$. Moreover, we let $b^*_{jk} = 0$ for all j and k. Then part (ii) of the definition of an equilibrium obviously holds. Part (i) holds for $j \neq i$, since $b_{jk} = b^*_{jk} = 0$ for all k is the only choice which satisfies (17). For person i, part (i) is satisfied since $\underline{x} = \underline{y}$ is i's optimal point.

Then assume that $\underline{x}$ is the optimal point for no individual. Condition (1) implies that the Pareto-optimal point $\underline{x}$ does not lie on the boundary of the feasible set. We can conclude that $\underline{x}$ is the maximum of a function v_Λ as given by (7). Then (8) or, equivalently, (21) follows. We define, for $i = 1,\ldots,N$,

$$(22)\quad A_i = \lambda_i^2 \cdot \sum_{k=1}^{K} \left(\frac{\partial v_i}{\partial x_k}\right)^2 ,$$

where the derivative is evaluated at $\underline{x}$. Each A_i is clearly non-negative. We know that $\lambda_i > 0$ for some i. The sum in the right-hand side of (22) is strictly positive; otherwise, $\underline{x}$ would be the maximum of v_i, a case we have ruled out. Hence $A_i > 0$ for same i. Then we define b^*_{ik} by

$$(23)\quad b^*_{ik} = \lambda_i \frac{\partial v_i}{\partial x_k} ,$$

for all i and k. It is easy to see that the definition of an equilibrium is satisfied: From (22) and (23), we conclude that $\underline{b}_{i.}^{*}$ satisfies (17) with equality; the discussion in connection with equation (20) verifies part (i) of the definition; and part (ii) follows from (21). This completes the proof. ||

Returning to the assumption that $A_1,\ldots,A_N$ are fixed, we have finally:

Proposition 4

There exists an equilibrium.

Sketch of Proof

The proof uses Kakutani's fixed-point theorem.[36] That is, we construct a correspondence ϕ which satisfies the premise of that theorem. This correspondence has a fixed point, which is then shown to be an equilibrium.

Let $\underline{x}$ be given. For each i, choose numbers $b_{i1},\ldots,b_{iK}$ which satisfy the following:

(i) If $\underline{x}$ is the maximum of v_i, let $b_{i1},\ldots,b_{iK}$ be any numbers which satisfy (17).

(ii) Otherwise, define b_{ik} by (20), where $\lambda_i \geq 0$ is chosen such that (17) holds with equality. (The derivative is evaluated at $\underline{x}$.)

When these numbers are chosen, compute $\underline{y}$ by (19). This $\underline{y}$ shall be an element of $\phi(\underline{x})$. The set $\phi(\underline{x})$ is constructed by choosing the numbers b_{ik} in every way consistent with the description above; for each choice, $\underline{y}$ is computed and included in the set.

As long as $\underline{x}$ is not the maximum of v_i, $\underline{b}_{i.}$ is uniquely determined by (ii). Moreover, in this case $\underline{b}_{i.}$ is a continuous function of $\underline{x}$. But

if $\underline{x}$ maximizes v_i and $A_i > 0$, there are infinitely many possible choices for $\underline{b}_{i.}$. If this is the case for any i, $\phi(\underline{x})$ will have more than one element.

For any $\underline{x}$, the set $\phi(\underline{x})$ is convex and compact. Moreover, ϕ is upper semicontinuous. It is possible, however, that some $\underline{y} \varepsilon \phi(\underline{x})$ lies outside the feasible set. Moreover, the set on which ϕ is defined is not compact.

To solve these problems, we observe that (17) can be used to find an upper bound for $\|\underline{y} - \underline{x}\|$, when $\underline{y} \varepsilon \phi(\underline{x})$. Let the compact and convex set S be defined by[37]

$$\underline{x} \ \varepsilon \ S \quad \text{if and only if} \quad 0 \leq x_k \leq x_k^* \quad \text{for all} \quad k = 1, \ldots, K,$$

where x_k^* is the number given by (2). Using the bound on $\|\underline{y} - \underline{x}\|$, we can construct a compact and convex set S' such that $S \subseteq S'$ and $\phi(\underline{x}) \subseteq S'$ for all $\underline{x} \ \varepsilon \ S$. Then we extend ϕ to S' in the following way:

For $\underline{x} \ \varepsilon \ S'$, define $\underline{x}' = (x_1', \ldots, x_K') \ \varepsilon \ S$ by[38]

$$x_k' = \begin{cases} 0 \text{ if } x_k < 0, \\ x_k \text{ if } 0 \leq x_k \leq x_k^*, \\ x_k^* \text{ if } x_k > x_k^*. \end{cases}$$

Then let

$$\phi(\underline{x}) = \phi(\underline{x}') .$$

Here $\underline{x}'$ is a continuous function of $\underline{x}$. Now ϕ is defined as a correspondence on S', and it satisfies all the conditions of the fixed-point theorem. Hence ϕ has a fixed point, that is, there exists a point $\underline{x} \ \varepsilon \ S'$ such that $\underline{x} \ \varepsilon \ \phi(\underline{x})$. From conditions (1) and (2), it can be proved that $\underline{x} \ \varepsilon \ \phi(\underline{x})$ is impossible if $\underline{x} \ \varepsilon \ S' \setminus S$; hence the fixed point must belong to S. From the definition of ϕ , it is now easy to prove that $\underline{x}$ is an equilibrium. ||

Conditions (1) and (2) are not used in the proofs of the first two propositions.[39] In the proof of Proposition 3, condition (1) is used to show that any Pareto-optimal point is interior to the feasible set. In the absence of (1), it will still be true that any interior Pareto-optimal decision can be obtained as an equilibrium.

The conditions are used in a more essential way in the proof of Proposition 4. Without (1) and (2), we can still construct the set S' and the correspondence ϕ, and ϕ will have a fixed point in S'. But the fixed point need not belong to S; instead, it may have been created when we extended ϕ from S to S'. Hence we cannot conclude that there exists an equilibrium.

In practice, it is not unlikely that (1) and (2) will hold for most individuals but not all. Then the existence of an equilibrium is likely to depend on the relative influence of these groups of individuals. Of course, if the persons for whom (1) or (2) fails are without influence, that is, if $A_i = 0$ for these i, no problems can occur. It seems reasonable to expect that the same is true if A_i is relatively small for these individuals. Only if the persons who want nothing at all or an unbounded amount of a certain public good are fairly influential compared to those whose preferences are bounded, can problems arise.[40]

5. Computation of Equilibria

When an equilibrium has been found, it represents a solution with several desirable properties, as described in Section 4. But how can an equilibrium be computed?

Problems arise on two levels. For one thing, even if the functions v_i were known, computing an equilibrium would not be a trivial matter. We will return to this problem shortly, and we will also describe an algorithm which works in a fairly general class of cases. But first we will discuss the other problem: If a procedure is designed which brings us to an equilibrium when people report their preferences correctly, can we be sure it will be in their interest to do so?

Misrepresentation of preferences. Proposition 1 guarantees that when an equilibrium has been reached, nobody can gain by making a unilateral change of response. The interpretation of this result is discussed in Section 3, immediately following the Proposition. The result only addresses what happens at an equilibrium. Here we ask the following question: Can an individual, by incorrect reporting of preferences, bring about a change in the equilibrium, and thereby get a better outcome than the one which would have resulted from correct reporting?

When the number of individuals is small, it is easy to construct examples in which some individual can gain in this way.[41] If the number of individuals is greater and everybody's influence is small, these kinds of examples can still be constructed, but we expect them to become rarer and more special.[42]

This issue relates to the sense in which our method solves the problem that preferences are unknown to the central authority. In our procedure, the equilibrium responses constitute a Nash equilibrium. Ideally, we would want a solution based on dominant strategies. That is, we want the following situation:

> For each i, a set S_i is specified; this is the set of possible strategies or actions individual i can choose. When each i has chosen a strategy, the procedure mechanically computes the decision. For each i and any possible v_i, there exists a strategy in S_i which is an optimal action for i no matter what all other individuals do. This strategy shall only depend on v_i and is said to be a dominant strategy (for v_i).

If such a procedure exists, there is no loss of generality in assuming that each S_i is the set of possible valuation functions and that the dominant strategy always is telling the truth.[43] Hence the procedure can be assumed to be a "direct mechanism," that is, people report their preferences and the result is computed directly from these. Our procedure is not a direct mechanism in this sense; the outcome of the procedure as described in Section 3 depends not only on individuals' responses but also on the announced tentative decision, and the equilibrium concept is based on this description.

The question, therefore, is the following: Does there exist procedures which admit dominant strategies and at the same time guarantee Pareto-optimal outcomes? The answer is yes. For each i, we can define a procedure which makes i a "dictator," that is, a procedure which chooses i's optimal point as the outcome, regardless of other people's preferences. These procedures obviously lead to Pareto-optimal outcomes, and correct reporting of preferences is a dominant strategy for everybody. These procedures divide influence in an extremely uneven way. Since our goal is to construct a procedure which can be adapted to various distributional objectives, we are not satisfied with this class of procedures. When there are two or more public goods, the "dictatorial" procedures are, however, the only ones which satisfy the stated conditions. Therefore, if we insist on a more even distribution of influence and on Pareto optimality, we cannot rely on dominant strategies. Based on experience from related models, this conclusion should come as no surprise.[44] To the best of our knowledge, however, the statement does not follow from published results. It is not difficult to prove, but the proof is long and not very interesting, and it is not given here.[45]

The conclusion, therefore, is that an "ideal" solution of the type described above does not exist. Hence the procedure constructed in this paper is as good a solution as one can expect to the problem of misrepresentation of preferences. Again, there is a parallel to private goods markets: The competitive equilibrium is consistent with self-interested behavior only to the extent people actually regard prices as fixed.

The question then is: How good is this solution in discouraging misrepresentation? Stated in another way: How likely is it that participants in a practical application of the procedure will be able to gain by misrepresentation of preferences?[46] In the end, this is an empirical question, and it depends on the particular procedure used to compute equilibria. To answer the question, we would have to construct an algorithm which solicits information and computes an equilibrium, and then try it and see how it works when people react to it.[47] Nevertheless, something can be said about the issue on theoretical grounds. In our judgment, the problem of misrepresentation is not likely to be serious, at least not when the number of participants in the decision is large. Successful strategic behavior requires a lot of information about other people's preferences, information unlikely to be available. Therefore, it seems reasonable to assume that given their ignorance the participants will, after all, find it advantageous to report their preferences correctly. Misrepresentation will be a gamble, and, most likely, an unfavorable one.[48]

Informational considerations. Now suppose that people will answer questions about their preferences correctly. In the design of a procedure which computes equilibria, an important practical issue is the amount of information people are asked to transmit to the central authority. We cannot simply ask them to report their valuation functions. Even if this generated no incentive to report incorrectly, there is far too much information involved.[49] If we could restrict the functions to some class indexed by a finite number of parameters, the situation would be different; then the values of the parameters could be transmitted.[50] We will not discuss this issue further; we hope it will be explored in future investigations.

The information problem also implies that we can never compute equilibria exactly; the most we can hope is to come close. The sense in which we come close is important. In particular, we want Propositions 2 and 3 to hold approximately. That is, the outcome must always be close to the Pareto frontier, and any point on this frontier must be close to an outcome for some distribution of influence.

Iterations towards the equilibrium. A natural proposal for a computational algorithm is the following: We start with a point $\underline{x}$, arbitrarily chosen (or chosen in a way not specified here). This is considered the "tentative decision" as described in Section 3. Individual responses are given by equation (9), and a new decision $\underline{y}$ is computed by equation (3). This $\underline{y}$ is then treated as the tentative decision in a new round, and so on. Hopefully, this process will converge to an equilibrium.

As far as the equilibrium concept is concerned, the scale of the numbers $A_1,\ldots,A_N$ (or, equivalently, $B_1,\ldots,B_N$) has no significance; only the relative size of the numbers matters. For the iteration procedure just described, however, the scale is important, since it corresponds to the step length of the algorithm. The step length issue creates a problem in designing the mechanism: If the step length is small, the procedure will move slowly and will require many steps to come close to an equilibrium (unless the starting point happens to be close). On the other hand, if the step length is too large, there is a risk that the procedure will repeatedly "overshoot" the equilibrium and hence never converge. Faced with this dilemma, one can try to vary the step length as circumstances change, as is done in the mechanism used for illustration below.

There is no realistic chance of refining the procedure so that it always converges.[51] This conclusion is related to the issue of stability of equilibria in an ordinary competitive economy, a complicated question which is discussed at length elsewhere, and thus not explored in detail here.[52]

An example. For illustrative purposes, we have constructed an iteration mechanism of the type outlined above. Applied to a number of cases where the preferences are chosen from a relatively general class of valuation functions, the mechanism seems to function satisfactorily.

The procedure works in the following way:[53]

(a) For each i, the maximum $\underline{x}^{(i)}$ of v_i is found. Each individual but i is asked to react to $\underline{x}^{(i)}$ as described in Section 3, and it is determined whether $\underline{x}^{(i)}$ is an equilibrium.[54]

(b) If no $\underline{x}^{(i)}$ is an equilibrium, a weighted average of $\underline{x}^{(1)},\ldots,\underline{x}^{(N)}$ is computed, $A_1,\ldots,A_N$ being used as weights. The iteration procedure is started at this point. As long as the movement produced by any two consecutive steps in the iteration procedure point approximately in the same direction, the step length is not changed.

(c) When the increment of two consecutive steps form an angle which exceeds a prescribed limit, the step length is reduced by a fixed factor, that is, $A_1,\ldots,A_N$ are scaled down.

(d) If the increment is less than a given (small) number times the current step length, we stop.

The idea is that as long as any two consecutive increments are parallel or almost so, we move steadily towards the equilibrium and should continue to move at unchanged speed. When this is not true, we may have "overshot" the equilibrium or be in the process of doing so, and a cut in the step length is in order.

For each step in the iteration, every individual must send a message to the center consisting of K real numbers. This should not cause great problems. The total amount of information to be transmitted depends, of course, on the number of steps, which is related to the accuracy required.

At no stage will a participant who is ignorant about other people's preferences have an obvious incentive to engage in strategic behavior. During each iteration (except the last one), we do not compute the final outcome, but rather the starting point for the rest of the procedure. In general, an individual will want this starting point to be the best possible decision (though exceptions may again occur if good information about other people's preferences is available). This objective is achieved by responding honestly all the time.[55]

This mechanism could be applied to any profile of preferences; sometimes it will converge and sometimes, we expect, it will not.[56] We have experimented with preferences from a class of valuation functions formed by analogy from the class of production functions with constant elasticity of substitution. For a given i, we let x_{i0} denote the amount of money person i can spend after taxes if public goods decision $\underline{x}$ is made. (This depends on $\underline{x}$, i and the predetermined tax system.) Then i's valuation function is given by[57]

$$v_i(\underline{x}) = (-\frac{1}{\rho})(\alpha_0 x_{i0}^{-\rho} + \alpha_1 x_1^{-\rho} + \dots + \alpha_K x_K^{-\rho}) ,$$

where ρ and $\alpha_0,\alpha_1,\dots,\alpha_K$ are parameters (which may vary from person to person). The numbers α_0 and $\alpha_1,\dots,\alpha_K$ are positive and represent the weights person i puts on private goods and the various public goods, while ρ is related to the "elasticity of substitution" and satisfies $\rho \neq 0$ and $\rho > -1$. When ρ tends to 0, v_i essentially converges to the Cobb-Douglas utility function, given by

$$v_i(\underline{x}) = \alpha_0 \ln x_{i0} + \alpha_1 \ln x_1 + \dots + \alpha_K \ln x_K .$$

We have applied the mechanism to a number of cases in which the preferences are of this type and the tax system is linear, that is, each person pays a predetermined share of public expenditure. We have tried the mechanism on a number of typical cases, and on a number of examples generated by choosing the parameters by lottery from a given probability distribution. In each of these cases, the mechanism succeeded in computing an equilibrium in a reasonable number of steps. Therefore, we feel confident that the procedure works for this fairly general class of preferences.[58]

6. Distributional Objectives

In the Introduction, we formulated two conditions the procedure should satisfy. Condition (a) requires that when individuals behave in a self-interested manner, the procedure assure Pareto optimality. Propositions 1, 2 and 4 guarantee this; an equilibrium will always exist, it is sustained when individuals act in a self-interested way, and it is Pareto optimal.[59]

Condition (b) requires that the procedure be able to accommodate different distributional objectives. Proposition 3 addresses this issue. It says that any point on the Pareto frontier can be obtained as an equilibrium of our procedure for an appropriate distribution of influence points. In a sense, this implies that the condition is satisfied. Some questions remain, however.

Multiple equilibria. Problems may arise if, for the same distribution of influence points, there are two or more possible equilibria. The specific method of computing outcomes will then decide which of the equilibria is reached. No matter how the computation is done, there will be at least one equilibrium which cannot be obtained using our procedure.[60]

Is this a serious problem? When there are only two participants, it is serious. Then the possibility of multiple equilibria makes the procedure trivial, in the following way: If one person has more influence points than the other, no matter how small the difference, the former is a "dictator" whose optimal point is always chosen. If we rule out dictatorial procedures, therefore, we must give the two individuals equal influence. But then all Pareto-optimal points will be equilibria. Hence the outcome will depend entirely on the specific procedure used for computing the equilibrium, and we have no opportunity for accommodating different distributional objectives.[61]

When there are three or more individuals, there is reason for optimism. For certain kinds of preferences, uniqueness of the equilibrium can be proved.[62] Examples can be constructed in which there are two or more equilibria, but these examples have the flavor of being "special cases." The issue is complicated, and we will not go into detail.[63]

Interpretation of the distribution of influence points. Distributional objectives can be formulated in many ways, both formal and informal. In order to apply our procedure, we must "translate" a given set of distributional objectives into an allocation $A_1,\ldots,A_N$ of influence points. To do this, we would like to have an answer to the converse question: What does it mean, in terms of some formally or informally expressed objectives, that people are given influence points $A_1,\ldots,A_N$?

Precise answers to these questions cannot be given. More influence points imply, in an intuitive sense, more influence. But how good the outcome is, in individual i's opinion, will depend on v_i as well as on A_i. If the preferences expressed by v_i are "average," the outcome may be very good even if A_i is small.

Suppose that our distributional objective is to treat everybody equally. The obvious way to achieve this is to let all the numbers $A_1,\ldots,A_N$ be equal. But even in this case, it is not clear in what sense equality will be achieved.

This problem of interpretation is illustrated by the fact that the equilibrium concept is not invariant under changes in the units of measurement of the public goods. That is, if these units are changed while everything else (including $A_1,\ldots,A_N$) is kept fixed, the equilibrium may change. Somebody will gain and somebody will lose from the changes in units of measurement, therefore, the numbers $A_1, \ldots, A_N$ do not fully describe the distribution of influence.

These problems are mitigated by the fact that when influence points are assigned, preferences are unknown. In this situation, it is reasonable to say that an equal number of influence points implies equal influence, and a higher number of points means more influence, recognizing that the interpretation of these concepts is not absolutely clear.

7. Applicability of the Model

The ultimate purpose of our analysis is normative: We want to contribute to the construction of procedures for practical decision making. The main subject of this paper, namely the formal construction of a procedure and description of its theoretical properties, is but one step towards this goal. Much work remains before it can be determined whether our procedure, in the present or some modified form, is suited for practical applications. In part, this work will take the form of trials and experiments.[64] We will not attempt an exhaustive discussion of implementation issues, but a few remarks should be made.

Much of the discussion which follows is concerned with application of our mechanism to decisions made by and for the society at large, through direct votes by the citizens or through legislative action. Realistically, however, we do not expect the procedure to be implemented soon in such situations. Other kinds of decision-making bodies are more likely to attempt to use formalized mechanisms of the type we describe. The discussion should be viewed in this light.

Resemblance to informal procedures. We believe that our procedure, to some extent, captures aspects of existing informal processes for selecting public goods. Frequently, one can observe that an assembly is reluctant to make decisions by majority vote even if its rules allow such decisions; the members prefer to debate and debate and debate until consensus is reached. Somehow, participants in such a procedure must decide whether to stick to their positions or to yield. Usually, this involves some judgment about the strength of other people's preferences.

We believe that decision making in university departments often takes this form. To see the analogy, consider a department meeting (or a series of meetings) in which decisions must be made about a number of issues, such as appointments, curriculum, allocation of funds among competing uses, etc. Endowments of influence points will correspond to power or prestige in the department. These are not specified exactly, but regular participants will usually have a fairly good idea about who the powerful members are. Discussions are held and a consensus emerges on most issues. Some may speak in favor of a greater share of funds for fellowships, others for less, while some stay completely out of the debate on this issue. Eventually, a balance of power and views is established, that is, a consensus is reached.

Casual observation suggests that something akin to the diminishing returns property of our square root formula may be in effect. A lengthier or more passionate speech carries more weight than a brief expression of opinion, but not nearly in proportion to the relative time or intensity expressed. No participant can get more total influence by speaking passionately on every issue; this will have the effect of depreciating this person's "currency."

To the extent that our procedure reflects aspects of existing, informal decision processes, it is more likely to be accepted and implemented. In this case, the procedure does not represent entirely new principles, instead, it is a formalization of familiar phenomena.

This is not to say that our procedure merely replicates existing processes. One important difference has to do with decision-making costs. In an informal process as described above, real resources have to be expended as the participants express the relative strength of their preference on the different issues. The cost, in the form of the time it takes to reach a consensus, is frequently non-trivial. Committees often spend days in debating even relatively unimportant issues. Our mechanism attempts to substitute a mechanical process for the tugging and hauling of the political arena, thus drastically reducing the time costs of group decision making. In our procedure, people express their preferences by expending artificial influence points. These are essentially costless to create and spend. Hence time and other real resources are saved.

Incentives for participation. When faced with a complicated decision problem, people will frequently have to spend significant amounts of time and effort to assess their own preferences accurately. In the model we describe, if there are many participants, any one of them can have only a small influence on the outcome. The question then is: Will the participants find it worthwhile to assess their preferences carefully and respond accurately? This problem exists in all types of collective decision making and voting, but it is likely to become more serious as the participants are supposed to provide more complicated information.

In situations where public goods decisions are made directly by thousands or millions of voters, the problem might well be a serious one. For decisions made by a smaller group or by an elected assembly, the situation is probably different. In the case of an elected assembly, the participants are specifically assigned the task of assessing their own and their constituents' preferences, and we can hope that this job will not be taken lightly.

The likelihood that people will abstain or be less than careful in computing their responses, will depend on how important the issues are. That is, this likelihood depends on how strongly they feel about the issues described by the model compared to the effort involved in assessing the preferences. This fact tends to mitigate the problem; when inaccurate reporting results in an outcome which is not a true Pareto optimum, the difference will usually be of little consequence to anyone.

The assumption of fixed tax schemes. Since the objective of our study is so clearly normative, one can ask why we have imposed the condition that financing mechanisms and tax schemes be fixed for any possible public goods decision (see condition (ii) in the Introduction). The condition describes reality, but an aspect of reality that should perhaps be changed.

If the condition is removed, it is possible to construct procedures whose outcomes are unconstrained Pareto optima and which otherwise have the same desirable properties as our procedure.[65] Proponents of systems which rely on monetary transfers to obtain correct reporting of preferences would point out that we, by imposing this condition, have deprived ourselves of the possibility of achieving true Pareto optimality. If the condition is removed, a procedure can be constructed which leads to an outcome preferred by everybody to the outcome of our procedure. This alternative procedure can also be adapted to various distributional objectives; hence distributional arguments cannot be used to defend our position.

We will respond with arguments of two types. First, we believe that the tradition that taxes shall not depend on how people vote is thoroughly established in modern democracies, and proposals to change this rule are not likely to be taken seriously. In particular, undertaking such a change would mean abolishing the secret ballot.[66] If one wants to contribute to the improvement of practical decision making, proposals which violate such fundamental traditions will simply not be helpful.[67]

Second, we believe that these traditions have a sound, substantive justification, based on fundamental notions of equity. By saying this we imply that there are relevant factors not captured by our model. Hence a Pareto improvement, in the sense of the model, is not necessarily an improvement for everybody when all factors are considered. In response to this, we may be challenged to extend the model and incorporate these "other factors." At least in the short run, this is not a practical option. One has to make an informal tradeoff between factors captured by the model and non-formalized ones. Having made this tradeoff, we conclude that tax systems, in many important cases, should be independent of individual votes, as assumed in this paper.

Allotment of influence points. In Section 6, we discussed the interpretation of the endowments of influence points. Here we address the issue of how this allocation may be determined.

First we should emphasize that nothing requires the numbers $A_1,\ldots,A_N$ to be public knowledge. If desirable, these numbers can be kept secret from everybody but the central authority administering the decision making.[68]

The endowments can be assigned by a deterministic procedure based on precise and "objective" criteria, or the influence points can be allotted at the discretion of an authorized leader or committee. In both cases, the basis for the allocation is supposed to be distributional objectives or moral claims on influence. Presumably, such criteria are exogenously given and cannot be changed by the individuals' own actions, at least not in the short run.

The objective can, for example, be to maintain the status quo. That is, assume that certain decisions have traditionally been made by some informal procedure, but it is decided to introduce a mechanism of the type described in this paper. For example, the purpose of this can be to save time and other decision-making costs. Presumably, one would not want this change to increase some individuals' influence at the expense of others; hence the allocation of influence points should reflect, as accurately as possible, the prevailing distribution of power.

As an alternative, the allocation can be based on criteria which can be influenced by the participants. That is, the influence points may, in whole or in part, be used to reward certain types of action. Again, the endowments can be determined by objective criteria or discretionary decisions. (To continue using university departments as an example, we could imagine professors receiving influence points on the basis of teaching load, or on the dean's discretionary evaluation of the quality of their work.) Casual observation suggests that in many organizations, the members' influence is partly determined by their contributions to the provision of public goods. New questions will arise if the influence points are allocated in this way; they will not be discussed here but may be the subject of future investigations.[69]

General principles for allocations of influence points can hardly be given. An appropriate procedure will have to be worked out in any specific situation where our mechanism might be implemented. But we suspect that the introduction of a formalized procedure like the one presented in this paper will often lead to increased use of formal rules and objective criteria elsewhere in the system. In most cases, it seems unlikely that somebody will be given the authority to allot influence points in a discretionary way. Especially in connection with legislative decisions and other cases in which people feel that fundamental rights and important interests are at stake, we believe that discretionary allocation of influence will not be accepted.

A great legislative leader, it is often asserted, can "test the waters," formulate a package of legislation which represents a workable compromise, and then use a variety of persuasive means to have the package accepted by the legislature (or at least by the leader's own party). In terms of our model, the leader's role can be interpreted in two different ways: First, the leader can be viewed simply as an agent who computes the outcome. The participants' relative influence is given, though not in a formalized manner. The leader's job is to assess people's power and preferences, and then seek out the "center of gravity" for the forces which pull in different directions. (The leader's own preferences may or may not be among these forces.) Second, the leader may also be the one who dispenses influence and decides upon the power of the representatives.

If the decision making is formalized, the first of these roles will be taken over by the procedure. One could imagine, however, that legislative leaders essentially keep their position as "power brokers" and continue to perform the second function: In much the same way as they used to hand out committee assignments, in our model they would allocate endowments of influence points on a discretionary basis. But we do not believe that most legislators, or their constituents, would accept such a process. Acceptance seems equally unlikely whether the leader's allocations are kept secret or made public. If our mechanism were put into effect, it would become more difficult to base differentials in power on discretion or nonobjective criteria, such as force of personality or strength of one's political network. We would predict, therefore, that legislators would get equal endowments of influence points, or that endowments would depend on observable criteria such as the size of the electorates, support in the most recent election, seniority, etc. To what extent such a development is desirable is a matter for commentators on and students of the political process to debate.

8. Concluding Remarks

Find a mechanism that assures the provision of an efficient bundle of public goods, when people's preferences are unknown to the authority which administers the decisions. This challenge, together with the constraint that the financing mechanism be externally imposed, was the motivation for this paper. We have shown that the challenge can be met, provided that individuals have convex preferences for possible public goods bundles and that individuals act in an uncoordinated, self-interested manner.

To recapitulate, the procedure works through a kind of weighted voting. Each participant has an exogenously given endowment of "influence points." The individual allots these points among the various goods, and uses them to "vote" for an increase or a decrease in the supply of each good. But influence points do not produce votes for movement on a linear basis; as a person spends more points on one good, the marginal value of an additional point decreases. Specifically, votes equal the square root of the number of points expended. This decreasing "productivity" of the influence points induces the participants to spread out their allocations, in a way which reveals their relative strength of preferences for changes in the provision of the different goods.

If the votes cast in favor of increasing the supply of each public good exactly balance the ones cast in the opposite direction, an equilibrium is achieved. The equilibrium is the outcome of the procedure; it represents an efficient bundle of public goods, as well as having other desirable properties which are detailed in Sections 3 and 4.

The distributional aspects of the outcome are determined by the endowments of influence points. By varying these endowments, the mechanism can be adapted to different sets of distributional objectives; see Proposition 3 and Section 6.

Problems exist in connection with the actual computation of an equilibrium, problems that parallel those that arise in relation to competitive markets for private goods. Computational issues are addressed in Section 5, in which we also present an algorithm which seems to be successful in dealing with a fairly general class of individual preferences.

The selection of public goods is an important assignment of governments. Decisions about public goods are also made by a variety of non-governmental entities; we have earlier used university departments as an example, but many others could be listed. The procedure by which such decisions are made is important for the substance of the decision and hence for the well-being of society or of the group involved. Traditional decision-making mechanisms have obvious shortcomings: Either they fail to take adequate account of the strength of people's preferences, or they render themselves vulnerable to strategic behavior. Our procedure tries to deal with both these problems.

We recognize that theoretical results like the ones presented in this paper only represent a first step. There is a long way to go before mechanisms of this type can become parts of procedures for real-life decision making. Still, the ultimate objective is practical applications; we hope that in time our work will contribute to improved decision making for a variety of organizations and polities.

FOOTNOTES

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[1]In the Lindahl solution, the public good is provided up to the point where marginal cost equals the sum of the participants' marginal valuations. The total cost is then divided among the participants in proportion to their marginal valuations at the chosen level.

[2]See, for example, Samuelson (1954), where the incentive problem is also discussed.

[3]The procedure which asks people to list their preferences but does not pay any attention to their answers, provides a trivial example; it generates no incentive for incorrect reporting. Non-trivial examples can also be constructed. In a somewhat different context, this is done by Zeckhauser (1973); see page 943. There the outcomes are lotteries over a finite set of alternatives. Procedures are constructed for which the unique best response is to report one's preferences correctly, but the resulting outcome is not Pareto-optimal.

[4]See note 59 below.

[5]Such procedures are reviewed in Groves (1979). We discuss the issue further in Section 7; see note 65 and accompanying text.

[6]For further discussion of "mechanism-constrained Pareto optimality," see Zeckhauser and Weinstein (1974), Section 3.

[7]A more satisfactory solution concept would have involved dominant strategies. This, however, is too much to ask for; see further discussion in Section 5.

[8]We shall later introduce assumptions which imply that only interior points of the feasible set can be chosen; hence we need the condition that the interior be non-empty. Note that when a set is convex and has a non-empty interior, every point in the set is the limit of points from its interior.

[9]This is not to say that there cannot be uncertainty concerning the public goods. Each potential decision x̲ may in itself be a risky prospect. This will cause no problems as long as the individual can rank these p[illegible]s and the ranking can be represented by a function v_i as described in the text. (For an example where such uncertainty clearly exists, let one public good be "expenditure on air pollution control." There is uncertainty both about the effectiveness of expenditures in reducing pollution, and about the ultimate effects of any given pollution level.) The assumption does rule out randomized decisions. That is, we will not allow the government's decision to be a probability distribution over public goods bundles, with the final outcome to be chosen subsequently by a lottery.

[10] The construction of such valuation functions and their properties are discussed by Zeckhauser and Weinstein (1974); see, in particular, their Section 4.

[11] We must not include in $\underline{x}$ so many aspects of the tax system that the set of technically feasible decisions has dimension less than K and therefore has an empty interior. In particular, total government expenditure should not be explicitly included as a coordinate.

[12] If total resources are fixed, we can include the wealth of each person but one in $\underline{x}$; see the previous note.

[13] Again, the vector $\underline{x}$ which describes a decision must be chosen in such a way that the feasible set has a non-empty interior. For example, if the decision essentially consists in dividing fixed resources among a number of activities, K must be one less than the number of activities.

[14] The issue is discussed after Proposition 4 in Section 4; see, in particular, note 40. Even in the cases where (1) and (2) are used, we do not claim that they represent the weakest possible conditions, but we will not go into the question of whether they can be relaxed. In the more general model mentioned in connection with note 8 where the set of possible decisions is compact and convex, the condition corresponding to (1) and (2) would require that at any point on the boundary of the feasible set, the gradient of v_i points inwards.

[15] See Zeckhauser and Weinstein (1974) for a more thorough discussion.

[16] We defer the question of what to do if $\underline{y}$ does not equal $\underline{x}$, so that $\underline{x}$ is not an equilibrium. At the moment we are concerned with the properties of the equilibrium concept; the existence of equilibria and ways to find them will be addressed later.

[17]One can ask whether f has to be the same function for all i and k; we comment on that question in note 31 below.

[18]Concavity of the functions v_i implies that the set $\{(u_1,\ldots,u_N) \mid$ there exists an $\underline{x}$ such that $u_i \leq v_i(\underline{x})$ for $i = 1,\ldots,N\}$ is convex. A standard application of the "supporting hyperplane theorem" proves the statement in the text.

[19]Consider, for example, the characterization given in Samuelson (1954). Since the cost of providing the public goods is incorporated into the functions v_i, marginal valuations must sum to 0 in our case, rather than summing to the marginal production cost for the public good. Other differences arise because we do not consider private goods and do not allow monetary transfers (except through the predetermined tax system). The weights $\lambda_1,\ldots,\lambda_N$ play a role similar to Samuelson's social welfare function. Specifically, λ_i corresponds to the derivative of the social welfare function with respect to i's utility, calculated at the final allocation.

[20]It is sufficient that the two sides of (9) be proportional. That is, (4) and (8) will coincide if $b_{ik} = \alpha\lambda_i \frac{\partial v_i}{\partial x_k}$, for all i and k and a fixed number $\alpha > 0$. But since the choice of the numbers λ_i is not unique, this formulation is no more general than (9).

[21]Information about absolute marginal valuation in the form of, for example, willingness-to-pay, is not necessary for mechanism-constrained Pareto optimality. In our model, it is not even meaningful to ask for such information.

[22]The latter assumption does not represent any real restriction, since a corner solution will lead to violation of (9); see the example below in the text.

[23]To prove (11), we form the Lagrangean expression of (5) and (10), with μ_i as the multiplier for the constraint (5). Setting the derivative with respect to a_{ik} equal to 0 and rearranging terms, we get (11). Note that σ_{ik} is the derivative of the function $|a_{ik}|$.

[24]The important point is not that the price is 1 but that the price system for any good is linear. The problems described below can be avoided neither by choosing a price different from 1, nor by letting the price depend on i and k.

[25]Strictly speaking, the issue is not the distance between the tentative decision $\underline{x}$ and the maximum of v_i, but the distance from this maximum to $\underline{z}^{(i)}$, that is, to the decision which would have been made if i did not participate. If the numbers $A_1,\ldots,A_N$ are relatively small, or if other people's responses more or less cancel each other out, these distances are approximately equal. In any case, the difficulty outlined in the text will exist. In formal terms, the difficulty is that the maximization problem given by (5) and (10) has a corner solution. In particular, (11) does not apply.

[26]This assumes that the marginal valuation is not 0.

[27]The assumption that marginal valuation is positive, is made only to simplify notation. Marginal valuation cannot really be constant, since v_i is strictly concave. But the assumption is a reasonable approximation, since the range over which person i can control the decision is likely to be small.

[28]It is not necessary to assume that the marginal valuation for each good is different from 0. As we shall see in the next section, the argument applies unless $\underline{y}$ is the maximum of v_i, that is, unless all marginal valuations are 0.

[29]A formal proof is given in Appendix A.

[30]This argument is correct only when $K \geq 3$; see Appendix A.

[31]This also answers the question in note 17: We can let f be different for different i, but there is no reason to do so, since we can vary A_i instead. Choosing different f for different k would, of course, violate (14).

[32]We should note that the problem discussed here, to reconcile individual self-interested behavior with a social objective (in our case Pareto optimality), is related to another problem about which there is a considerable literature: Suppose an expert has information about the probability distribution for some future event. The expert is asked to provide probability estimates, but cannot be forced to do so truthfully. How should the expert be rewarded, as a function of the estimate and the final outcome, so that giving the honest estimates is an individually rational action? This problem has several solutions, one of which (called the "spherical scoring system") strongly resembles the solution to our problem. See, for example, Shuford et al. (1966) and Winkler (1969).

[33]The public goods space can be of any of the types mentioned in Section 2. For one thing, it is possible that the public goods are non-monetary in nature or that no taxes are involved; in this case the functions v_i represent direct preferences for public goods. Moreover, it is possible that the complete model includes private and public goods, as well as a pre-determined tax system; then v_i is derived from the tax scheme and i's preferences for both classes of goods. If we assume a linear tax system, the given v_i can be derived from preferences for private and public goods which satisfy all the standard assumptions of economic theory, but these preferences do not have a simple mathematical form.

[34]It is assumed here that y is a feasible solution. In order for equation (20) to apply if y lies on the boundary of the feasible set, we must assume that v_i is defined, differentiable and concave (and that it represents person i's preferences) somewhat beyond this boundary. To be precise, v_i must be defined and have these properties on an open set which contains the set of feasible points. Together with condition (1), this implies that a move out of the feasible set is not wanted by any individual. This may seem reasonable. On the other hand, one can imagine people who would prefer the "state of limbo" which occurs if a y outside the feasible set is chosen. We ignore this possibility; in order to take account of it, we would have to describe what actually happens if the decision procedure makes an infeasible "decision." Note that conditions (1) and (2) are not necessary for (20) to apply.

[35]In Section 5, we shall discuss other forms of strategic behavior. There the number of participants may make a difference.

[36]See, for example, Debreu (1959), page 26.

[37]In the more general model, where the set of feasible decisions is compact and convex with non-empty interior (see Section 2 and note 8), we can let S be the feasible set.

[38]Alternatively, x' can be described as the point in S which is closest to x. When S is compact and convex, this is well defined and unique. This definition can also be used in the more general model.

[39]The proof of Proposition 2 depends, however, on the assumption made in note 34 concerning preferences for infeasible alternatives.

[40]Ideally, we would want a procedure which does not depend on (1) and (2). The desired procedure would coincide with the one constructed in this paper on the interior of the feasible set. Moreover, it would allow Pareto-optimal points on the boundary to be obtained as equilibria for appropriately chosen distributions of influence. We have not succeeded in constructing such a procedure; straightforward attempts fail to satisfy either Proposition 1 or Proposition 2.

[41]To construct an example, we use the type of preferences described in the last part of Section 3. That is, person i has optimal point $\underline{x}^{(i)}$, and v_i is defined by $v_i(\underline{x}) = -\|\underline{x} - \underline{x}^{(i)}\|^2$. Assume that there are three individuals with equal influence and two public goods. Then there will be a unique equilibrium. If the three optimal points form a triangle in which one angle is greater than or equal to 120°, the equilibrium will fall at the corner of this obtuse angle. Now suppose that $\underline{x}^{(1)} = (1,4)$, $\underline{x}^{(2)} = (3,1)$ and $\underline{x}^{(3)} = (6,1)$. The angle at $\underline{x}^{(2)}$ exceeds 120°, and $\underline{x}^{(2)}$ is the equilibrium. If individual 1 reports preferences with optimal point $\underline{y}^{(1)} = (3.5, 1.5)$, then $\underline{y}^{(1)}$ will be the equilibrium. The distance from $\underline{x}^{(1)}$ to $\underline{y}^{(1)}$ is smaller than the distance from $\underline{x}^{(1)}$ to $\underline{x}^{(2)}$; hence the incorrect reporting has been advantageous. (Reporting $\underline{y}^{(1)}$ is not person 1's optimal response, it is just one way of doing better than telling the truth.) This example is robust to small perturbations of the data.

[42]This is a conjecture based on experience from related problems; it is not a theorem we have proved. To make the statement precise, one can assume that people's preferences are drawn from a given probability distribution. For any N and any given distribution of influence, there will then be

a well-defined probability that the preferences are such that somebody can gain by misrepresentation. The claim is that this probability goes to 0 when N goes to infinity, provided that the distributions of influence for the different N are chosen such that everybody's relative influence goes to 0 when N increases. (This includes the possibility that influence is equal for every N.) If one person's share of the total influence remains large, there is no reason to expect the probability to vanish as N goes to infinity; this case can be considered the equivalent of monopoly.

[43]That is, suppose that a procedure with these properties is given and that corresponding dominant strategies are chosen. Then we can construct another procedure for which telling the truth is always a dominant strategy, such that the two procedures give the same outcome for any set of individual preferences, provided that everybody uses the dominant strategies mentioned. See Dasgupta, Hammond, and Maskin (1979), Theorem 4.1.1. In practical applications, there may be reasons for choosing other strategy sets than the set of possible valuation functions; for example, other strategy sets may reduce the amount of information individuals will have to transmit to the central authority. But on the theoretical level, if the solution concept is dominant strategies, there is no reason to consider other sets of strategies.

[44]A fundamental result in this area is the one proved by Gibbard (1973). For a similar result in a model more closely related to ours, see Green and Laffont (1977), Theorem 7.

[45]If there is only one public good, our assumptions imply that preferences are "single peaked." Then weighted majority vote is non-dictatorial and satisfies the conditions mentioned in the text, provided the weights

are such that ties never occur. (In this case, the procedure presented in this paper is just a complicated way of implementing weighted majority voting.) For $K \geq 2$, the impossibility result obtains. If preferences are restricted to the type described in note 41, the same is true for $K \geq 3$. (The proof of this is quite complicated.) But with $K = 2$ and preferences restricted this way, issue by issue weighted majority vote is non-dictatorial and satisfies the other conditions. (Again it is assumed that the weights are such that ties are impossible; this holds, for example, if weights are equal and N is odd.)

[46]Problems arise as soon as people try to gain by misrepresenting their preferences, even if they actually are hurt by their actions. Whenever somebody reports preferences incorrectly, there is no way the procedure can guarantee Pareto optimality. Even if we succeed in constructing a procedure in which advantageous misrepresentation is demonstrably impossible, there is a danger that people will not believe this and will try to "outsmart" the system. (This problem persists even if the procedure admits dominant strategies, as in the discussion above of the "ideal" solution.)

[47]Experiments have been carried out in connection with a procedure related to ours, namely the one presented by Groves and Ledyard (1977, 1978). The results seem encouraging; see Smith (1975), as cited by Groves and Ledyard (1977), footnote 2. We find this experience significant, because of certain similarities between our procedure and the one of Groves and Ledyard: The solution concept is of the same nature in the two models; in both places the solution is a Nash equilibrium. Also, there is a resemblance between the

formulas defining the two procedures, in both cases they are based on quadratic functions. As far as we can see, however, neither of the procedures can be obtained as a special case of the other. The models differ in the following ways: Groves and Ledyard consider private as well as public goods. They allow monetary transfers, that is, people's taxes may depend on how they vote. Such transfers are ruled out in our model. An equilibrium in their model is Pareto optimal in the strict sense; we achieve optimality subject to the imposed institutional constraints.

[48]In Hylland and Zeckhauser (1979), we discuss the same issue for a somewhat different model. See pp. 307-308, where we consider the difference between cases with small and large numbers of participants.

[49]To be precise, the information contained in a function v_i is equivalent to that of a countably infinite set of real numbers. For example, the function can be specified by giving its values on all rational points; this is enough since the function is assumed to be continuous.

[50]Examples of such classes are the set of quadratic functions (which includes, but is more general than, the class discussed in note 41), and the class of "constant elasticity of substitution" valuation functions, see below in the text.

[51]There do, however, exist procedures which can be used to compute equilibria and which always work, but these are not of the simple iterative form; see Scarf (1973).

[52]For a review see, for example, Quirk and Saposnik (1968), Chapter 5.

[53]The procedure is described in detail in Appendix B.

[54]This step is necessary, since it is possible (and indeed quite likely) that some $\underline{x}^{(i)}$ is an equilibrium, in which case the procedure described by (b) - (d) is unlikely to converge.

[55]One might suspect that such an argument could be made about any procedure, and that it is therefore essentially void. This is not the case. The point is that the mechanism responds positively to everybody's action at each stage, and this is necessary for the argument to apply. Suppose that we tried to speed up convergence by using a second-order procedure. That is, we would measure people's responses at two points and extrapolate from these data to a point where the responses sum to zero; this point will be the next approximation to the equilibrium. Such a procedure will react negatively to some of the responses and thus provide an obvious incentive to misrepresent one's preferences.

[56]Presumably, examples for which the mechanism does not converge can be constructed by analogy from similar examples concerning competitive markets for private goods. See Quirk and Saposnik (1968), Section 5.8, and Scarf (1960).

[57]The functional form given here is a strictly increasing transformation of a production function with constant elasticity of substitution; the transformation is chosen so as to make v_i strictly concave.

[58]See Appendix B for a more detailed account of the results.

[59]In this respect, our procedure contrasts with procedures based on majority vote. In discussing this, we assume, for simplicity, that N is odd. We say that $\underline{x}$ is a multi-dimensional median if any proposal to change

one number x_k while keeping the others fixed, will be voted down. Formally, this means that $\# \{i | v_i(\underline{x}) > v_i(\underline{y})\} > N/2$ whenever $\underline{x}$ and $\underline{y}$ differ on exactly one coordinate. This is a natural equilibrium concept for majority vote, and concavity of the functions v_i implies that a multi-dimensional median always exists. (A stronger concept is obtained by requiring that $\underline{x}$ get a majority over any $\underline{y} \neq \underline{x}$. Equilibria in this sense will generally not exist.) At the median, if a proposal is made to change one coordinate of $\underline{x}$, self-interested individuals will vote according to their true preferences. Hence we have a Nash equilibrium result, corresponding to Proposition 1. For $K = 2$, the multi-dimensional median is Pareto optimal; for $K \geq 3$, examples can be found in which this is not true. For details, see Zeckhauser and Weinstein (1974), Section 7.2. For the case $K = 2$, that paper contains an incorrect claim. It is asserted that an extra assumption, called first-order preferential independence, is necessary to prove that the median is Pareto optimal. In fact, Pareto optimality can be proved without using this condition. The example used to support the claim (Figure D on page 663), violates the rest of the assumptions; it uses preferences which are inconsistent with concave valuation functions.

[60] The computation method need not be invariant under changes of scale in $A_1,\ldots,A_N$. It is theoretically possible that we could construct a procedure in which all equilibria can be obtained by varying the scale of these numbers, but we doubt that a workable procedure of this type exists.

[61] Again, we ignore the theoretical possibility outlined in the previous note.

[62]Consider, for example, preferences of the type described in note 41. In this case, multiple equilibria can only occur if the optimal points of all individuals with positive influence lie on a straight line. When $K \geq 2$ and $N \geq 3$, this is essentially impossible.

[63]Again, an analogy can be drawn with the theory of the competitive market. See, for example, the discussion in Quirk and Saposnik (1968), Sections 3.7 and 5.6.

[64]For example, empirical studies are needed in order to construct a method for computing equilibria; see Section 5.

[65]A review of such procedures is given in Groves (1979), Section 5. In particular, we refer to the seminal works of Groves and Ledyard (1977, 1978). In their procedure, everybody's taxes are allowed to depend on everybody's reported preferences. Unconstrained Pareto optimality can be achieved even if we require that individual i's taxes do not depend directly on i's own reported preferences, but only on the outcome and on the preferences reported by other people. Such a procedure is constructed by Walker; see description in Groves (1979), page 236.

[66]Individual ballots need not be made public, but they would have to be signed and made available to the tax collector.

[67]This argument applies to decisions made by direct vote by the citizens. For decisions made by elected legislative bodies, the reference to the secret ballot is irrelevant, but still we believe that few will accept a tax system which lets the constituents' taxes depend directly on how their representative

and other legislators vote. We do not deny that situations may exist in which it is perfectly acceptable that taxes depend on people's votes; our claim is that the opposite is normally the case.

[68]It might appear that individual i must know A_i; this number does not, however, convey any information to somebody who does not know A_j for any $j \neq i$. In any case, person i could be instructed to assume $A_i = 1$; the central mechanism can rescale $\underline{b}_i$. appropriately.

[69]In particular, a discussion of whether the procedure guarantees Pareto optimality and non-strategic behavior would have to take account of the activity for which rewards are given and people's preferences concerning this activity.

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SUMMARY OF APPENDICES

Appendix A

We prove that the procedure constructed in Sections 3 and 4 essentially is the only solution to the problem posed there. Two results are proved, corresponding to the two ways of describing the procedure. If people are asked to submit their "votes for movement" directly (as described in Section 4), the uniqueness result holds when there are at least two public goods. In the formulation where "influence points" are used (see Section 3), the square-root function represents the only solution provided that $K \geq 3$.

Appendix B

We report in detail on the computational experiments mentioned in Section 5. The class of preferences which is used in the examples is described and the algorithm is presented. The examples consist of some "typical cases," as well as a number of cases where the parameters are randomly generated according to specified probability distributions. When a fairly high degree of accuracy is required, the latter cases needed, on the average, 17 iterations to reach an equilibrium. (Each iteration asks for individual responses twice.) In more than half of the cases, one person's optimal point is an equilibrium and no iteration is needed. If these cases are excluded, the average number of iterations is 37.

APPENDIX A

In this appendix, we prove that the square-root formula presented in Section 3 is essentially the only solution to the problem posed there. The appendix is not self-contained; it relies on concepts and notation developed in Sections 3 and 4. Since we now consider the responses of only one individual, we can simplify notation by dropping the index which refers to individuals; hence we write, for example, $\underline{a} = (a_1,\ldots,a_k,\ldots,a_K)$ instead of $\underline{a}_{i.} = (a_{i1},\ldots,a_{ik},\ldots,a_{iK})$.

Direct Choice of the b-vector

As pointed out in Section 4, the function f and the numbers a_k are used in Section 3 only to facilitate intuitive understanding; they play no essential role in the formal model. Hence we start by considering the case in which the individual chooses a vector $\underline{b} = (b_1,\ldots,b_K)$ directly. This vector shall be chosen from a given set which will be denoted S. This set represents the "rules of the game," and our task is to specify the set S.

When S is given, the self-interested individual solves the following problem:

(A1) Choose $\underline{b}$ to maximize $v(\underline{z} + \underline{b})$, subject to $\underline{b} \in S$,

where v is the continuously differentiable and strictly concave valuation function, and $\underline{z}$ is a fixed vector, representing the decision which would have been made if this individual had not taken part.

We must assume that (A1) always has a solution; otherwise, the whole discussion will not make much sense. One way to guarantee this, is to require that S be compact. An explicit assumption of this type is not necessary; see comments after the statement of Proposition A1. We will assume that S is convex. This assumption will simplify the argument but is not essential; we discuss it after we have proved the proposition.

If $\underline{b}^*$ is a solution to (A1), let $\underline{x} = \underline{z} + \underline{b}^*$ be the final outcome and define $\underline{c} = (c_1,\ldots,c_K)$ by

$$c_k = \left.\frac{\partial v}{\partial x_k}\right|_{\underline{x}} \qquad \text{for } k = 1,\ldots,K. \tag{A2}$$

Then consider the problem:

(A3) Choose $\underline{b}$ to maximize $\sum_{k=1}^{K} b_k c_k$, subject to $\underline{b} \in S$.

We shall show that $\underline{b}^*$ is also a solution to this problem: If it is not, there exists a $\underline{b}^{**} \in S$ such that $\sum_{k=1}^{K} (b_k^{**} - b_k^*)c_k > 0$. By convexity, $\underline{b}^* + \lambda(\underline{b}^{**} - \underline{b}^*) \in S$ for $0 < \lambda < 1$. The definition of $\underline{c}$ implies that $v(\underline{z} + \underline{b}^* + \lambda(\underline{b}^{**} - \underline{b}^*)) > v(\underline{z} + \underline{b}^*)$ for sufficiently small $\lambda > 0$; hence $\underline{b}^*$ does not solve (A1), and the claim is proved. Conversely, if $\underline{b}^* \in S$ is not a solution to (A1), and $\underline{x}$ and $\underline{c}$ are defined as above, then $\underline{b}^*$ does not solve (A3). This is easily proved by using the fact that v is concave.

Therefore, we can concentrate on maximization problems of the type (A3). Our objective is to choose S such that the self-interested individual will choose a vector $\underline{b}$ which is parallel to $\underline{c}$; see equation (9) in Section 3. This must be true for any $\underline{c}$ which can occur. But it is easy to see that $\underline{c}$ can be any K-dimensional vector. Now it is clear what we want to prove.

<u>Proposition A1</u>: Let $K \geq 2$ be an integer, and let S be a convex subset of the K-dimensional Euclidean space. Assume that for any K-dimensional vector $\underline{c}$, there exists a $\underline{b}^*$ which is a solution to (A3) and which satisfies

$$\underline{b}^* = \alpha\underline{c}\,, \tag{A4}$$

for some number $\alpha > 0$. (The number α may depend on $\underline{c}$.)

Then there exists a number $A > 0$ such that

$$S = \{\underline{b} = (b_1,\ldots,b_K) \mid \sum_{k=1}^{K} b_k^2 \leq A\}\,. \tag{A5}$$

It is a part of the premise that (A3) always has a solution of the special type described by (A4). Hence it is implicitly assumed that S is such that (A3) has a solution for any $\underline{c}$, and no explicit assumption is necessary to assure this.

The proposition is not true for $K = 1$. Then any set of the form $S = [-B_1, B_2]$, where B_1 and B_2 are positive numbers, satisfies the premise. But if $B_1 \neq B_2$, the conclusion does not hold.

By applying standard techniques for solving constrained maximization problems, we can prove that a set S given by (A5) will satisfy the premise of the proposition, that is, the converse of Proposition A1 holds.

For a given $\underline{c}$, the problem (A3) may have many solutions. The premise only requires that at least one of these be parallel to $\underline{c}$. Let $\underline{0}$ denote the zero vector $(0,\ldots,0)$. For $\underline{c} = \underline{0}$, it is clear that only $\underline{b}^* = \underline{0}$ can satisfy (A4), but α can have any positive value. As a first step towards proving the proposition, we shall show that for $\underline{c} \neq \underline{0}$, the number α of (A4) is unique. Then $\underline{b}^*$ is also unique.

Suppose that this claim is false. Then there exist vectors $\underline{c}$, $\underline{b}^*$ and $\underline{b}^{**}$, and positive numbers α and α', such that both $\underline{b}^*$ and $\underline{b}^{**}$ are elements of S and are solutions to (A3), $\underline{c} \neq \underline{0}$, $\underline{b}^* = \alpha\underline{c}$, $\underline{b}^{**} = \alpha'\underline{c}$ and $\alpha \neq \alpha'$. Without loss of generality, we can assume $\alpha > \alpha'$. Then

$$\sum_{k=1}^{K} b_k^{**} c_k = \alpha' \sum_{k=1}^{K} c_k^2 < \alpha \sum_{k=1}^{K} c_k^2 = \sum_{k=1}^{K} b_k^* c_k ,$$

contradicting the assumption that $\underline{b}^{**}$ is a solution to (A3). The claim is proved.

Hence we can define a function $\alpha(c_1,\ldots,c_K)$, whose value is the α of (A4). This is defined when $c_k \neq 0$ for at least one k, and $\alpha(c_1,\ldots,c_K) > 0$ whenever it is defined. The optimal value of (A3) is equal to

$$\alpha(c_1,\ldots,c_K)(c_1^2 + \ldots + c_K^2) . \tag{A6}$$

Now let a K-dimensional vector $\underline{d} \neq \underline{0}$ be given, and let $\underline{b} \in S$ be a vector which solves (A3) and satisfies (A4) when $\underline{d}$ is substituted for $\underline{c}$. Then we will have $b_k = \alpha(d_1,\ldots,d_K)\cdot d_k$ for all k. This $\underline{b}$ can be tried as a solution to the original (A3), the value of which then becomes

$$\alpha(d_1,\ldots,d_K)(c_1 d_1 + \ldots + c_K d_K) .$$

This can obviously not give a value of (A3) which exceeds the optimal value given by (A6). Hence

$$\alpha(c_1,\ldots,c_K)(c_1^2 + \ldots + c_K^2) \geq \alpha(d_1,\ldots,d_K)(c_1 d_1 + \ldots + c_K d_K) . \tag{A7}$$

By interchanging the roles of c_k and d_k, we also get

$$\alpha(d_1,\ldots,d_K)(d_1^2 + \ldots + d_K^2) \geq \alpha(c_1,\ldots,c_K)(c_1 d_1 + \ldots + c_K d_K) . \tag{A8}$$

Now we fix the numbers c_k for $k = 2,\ldots,K$ such that $c_k \neq 0$ for at least one $k \geq 2$. Then α can be viewed as a function of one variable; that is, $\alpha(c_1,\ldots,c_K) = \alpha(c_1)$. We set $c_1 = c$, $d_1 = c + \varepsilon$ for some $\varepsilon > 0$, and $d_k = c_k$

for k = 2,...,K. For simplicity, we write $C = \sum_{k=2}^{K} c_k^2$; then C is fixed and strictly positive. Now (A7) and (A8) give

(A9) $$\alpha(c)(c^2 + C) \geq \alpha(c + \varepsilon)(c(c + \varepsilon) + C)$$

and

(A10) $$\alpha(c + \varepsilon)((c + \varepsilon)^2 + C) \geq \alpha(c)(c(c + \varepsilon) + C) \ .$$

This implies

(A11) $$\alpha(c+\varepsilon) \frac{c}{c^2+C} \varepsilon \ \leq \ \alpha(c) - \alpha(c+\varepsilon) \ \leq \ \alpha(c+\varepsilon) \frac{c+\varepsilon}{c(c+\varepsilon)+C} \varepsilon \ .$$

For a moment, we restrict our attention to positive values of c. Let a closed interval $[\underline{c},\bar{c}]$ be given, where $0 < \underline{c} < 1 < \bar{c} < \infty$. From (A11), we conclude that α is strictly decreasing for $c > 0$; hence it is bounded by $\alpha(\underline{c})$ on $[\underline{c},\bar{c}]$. The expression $(c+\varepsilon)/[c(c+\varepsilon)+C]$ is also bounded, when c and ε are such that $\underline{c} \leq c$ and $c+\varepsilon \leq \bar{c}$. This implies that α is continuous on $[\underline{c},\bar{c}]$.

We shall now prove that if $\alpha(\bar{c})$ is given, (A11) uniquely determines α. To be precise, let β be a function which is defined and strictly positive on $[\underline{c},\bar{c}]$, assume that (A11) holds with β substituted for α for all c and ε with $\underline{c} \leq c < c+\varepsilon \leq \bar{c}$, and assume $\alpha(\bar{c}) = \beta(\bar{c})$. Then we shall prove that $\alpha(c) = \beta(c)$ for all $c \in [\underline{c},\bar{c}]$. By an argument used above, we can conclude that β is continuous and strictly decreasing. Suppose that $\alpha(c) > \beta(c)$ for some $c \in [\underline{c},\bar{c}]$. (The situation is symmetrical; hence the case $\alpha(c) < \beta(c)$ is treated similarly.) From earlier remarks, including the assumption that β is strictly positive, we can conclude that the function $\alpha(c)/\beta(c)$ is continuous and has a maximum on $[\underline{c},\bar{c}]$. Let the maximum be attained at c^*. The maximum must exceed 1, and since $\alpha(\bar{c}) = \beta(\bar{c}) > 0$, $\alpha(c^*) > \alpha(\bar{c})$ and $\beta(c^*) > \beta(\bar{c})$, we can find a number η such that

$$\text{(A12)} \qquad \frac{\alpha(c^*) - \alpha(\bar{c})}{\beta(c^*) - \beta(\bar{c})} > \eta > \frac{\alpha(c^*)}{\beta(c^*)} > 1 \quad .$$

By using the right-hand inequality in (A11) for α and the left-hand inequality for β, we get

$$\text{(A13)} \qquad \frac{\alpha(c) - \alpha(c+\varepsilon)}{\beta(c) - \beta(c+\varepsilon)} \leq \frac{\alpha(c+\varepsilon)}{\beta(c+\varepsilon)} \cdot \frac{(c+\varepsilon)(c^2+C)}{c(c(c+\varepsilon) + C)} \quad ,$$

for all c and $\varepsilon > 0$ where c and $c+\varepsilon$ belong to $[\underline{c},\bar{c}]$. The last term in the right-hand side of (A13) exceeds 1 and is bounded by $(c+\varepsilon)/c \leq 1 + \varepsilon/\underline{c}$. Hence this term tends to 1 as ε tends to 0, uniformly in c. By combining this with (A12) and the definition of c^*, we can find $\varepsilon_0 > 0$ such that, whenever $0 < \varepsilon < \varepsilon_0$,

$$\text{(A14)} \qquad \frac{\alpha(c) - \alpha(c+\varepsilon)}{\beta(c) - \beta(c+\varepsilon)} < \eta \quad .$$

Now we choose an increasing sequence of numbers $c^{(0)},\ldots,c^{(m)},\ldots,c^{(M)}$, where $c^{(0)} = c^*$, $c^{(M)} = \bar{c}$, and $c^{(m)} - c^{(m-1)} < \varepsilon_0$ for $m = 1,\ldots,M$. For each m, (A14) implies

$$\alpha(c^{(m-1)}) - \alpha(c^{(m)}) < \eta(\beta(c^{(m-1)}) - \beta(c^{(m)})) \ .$$

Adding this for $m = 1,\ldots,M$, gives

$$\alpha(c^*) - \alpha(\bar{c}) < \eta(\beta(c^*) - \beta(\bar{c})) \ ,$$

which contradicts (A12). This completes the proof that $\alpha(c) = \beta(c)$ for all $c \in [\underline{c},\bar{c}]$.

Now we must find a function β which satisfies (A11) and has $\beta(c) > 0$ for $c > 0$. We can do this by using the fact that the converse of the proposition is true. Let S be given by (A5) and let $B = \sqrt{A}$. For $\underline{c} \neq \underline{0}$, it is easily seen that the unique solution to (A3) is $\underline{b}^*$ given by $b_k^* = c_k B/\sqrt{c_1^2+\ldots+c_K^2}$. The corresponding number α of (A4) is equal to:

(A15) $$\beta(c) = \frac{B}{\sqrt{c^2+C}} \quad .$$

Earlier arguments prove that the premise of the proposition implies (A11). Hence we can conclude that the function β given by (A15) satisfies (A11). This holds for any $B > 0$. (The claim that β satisfies (A11) could also have been proved directly, by straightforward but tedious algebra.)

The function β is positive for $c > 0$, and by choosing B appropriately, we get $\alpha(\bar{c}) = \beta(\bar{c})$. By the uniqueness result proved above, $\alpha(c) = \beta(c)$ for all $c \in [\underline{c},\bar{c}]$. In particular, this holds for $c = 1$, which implies $B = \alpha(1)\cdot\sqrt{1+C}$. That is, B does not depend on $\underline{c}$ or $\bar{c}$. For any $c > 0$, we can choose $\underline{c}$ and $\bar{c}$ such that $\underline{c} < c < \bar{c}$; therefore, $\alpha(c) = \beta(c)$ for all $c > 0$ with B as specified above.

We can now return to (A11) and consider the function α for negative values of the variable. Here (A11) implies that α is a strictly increasing function. An argument similar to the one used above proves that $\alpha(c) = \beta(c)$ for $c < 0$, when β is given by (A15), with $B = \alpha(-1)\cdot\sqrt{1+C}$.

Since α is monotone for $c > 0$ and for $c < 0$, the limits $\lim_{c\to 0^+} \alpha(c)$ and $\lim_{c\to 0^-} \alpha(c)$ exist. In (A9) and (A10) we let $c = 0$ and let ε go to 0 from above. The equations then imply $\alpha(0) = \lim_{c\to 0^+} \alpha(c)$. By setting $\varepsilon = -c$ and letting c tend to 0 from below, we similarly get $\alpha(0) = \lim_{c\to 0^-} \alpha(c)$. Then we can conclude that the number B in (A15) is the same for $c > 0$ and for $c < 0$. Hence $\alpha(c) = \beta(c)$ for all c, where β is given by (A15), with a fixed number $B > 0$.

We return to regarding α as a function of K variables. We have proved the following: For fixed but arbitrary numbers $c_2,\ldots,c_K$, at least one of which is non-zero, and for all c_1:

$$\text{(A16)} \qquad \alpha(c_1,\ldots,c_K) = \frac{B}{\sqrt{c_1^2 + \ldots + c_K^2}} \quad ,$$

where B can depend on $c_2,\ldots,c_K$. Of course, the entire argument can be carried through with the k'th coordinate playing the role the first coordinate played in the argument above. That is, for fixed numbers $c_1,\ldots,c_{k-1},c_{k+1},\ldots,c_K$ which are not all 0 and for all c_k, (A16) holds. Here B can depend on k and $c_1,\ldots,c_{k-1},c_{k+1},\ldots,c_K$. It remains to be proved that B is, in fact, the same number in all cases.

For any $\underline{c} \neq \underline{0}$, the B for which (A16) holds is unique. In particular, in order for (A16) to be true for $(1,\ldots,1)$, we must have $B = \alpha(1,\ldots,)\cdot\sqrt{K}$. We fix B at this value. Now let any $\underline{c} \neq \underline{0}$ be given, and choose k such that $c_k \neq 0$. We shall move from $(1,\ldots,1)$ to $\underline{c}$ in K steps. First we change the k'th coordinate from 1 to c_k. Then we make similar changes in the other coordinates, changing one of them at a time in an arbitrary order. Each step is of the type described above; therefore, the truth of (A16) is preserved. Hence (A16) holds for all $\underline{c} \neq \underline{0}$ with the fixed $B = \alpha(1,\ldots,1)\cdot\sqrt{K}$.

Now we let $A = B^2$. Choose any vector $\underline{c} = (c_1,\ldots,c_K)$ with Euclidean norm (or length) equal to B, that is, assume $\sum_{k=1}^{K} c_k^2 = A$. Then $\alpha(c_1,\ldots,c_K) = 1$, and the vector $\underline{b}^*$ of (A4) is equal to $\underline{c}$. Since $\underline{b}^*$ is a solution to (A3), it belongs to S. This proves that S contains all K-dimensional vectors with norm equal to B. If $\underline{b}$ has norm less than B, there exists a $\underline{c}$ such that $||\underline{c}|| = B$ and $\underline{b} = \lambda\underline{c}$ with $0 \leq \lambda < 1$. (If $\underline{b} = \underline{0}$, any vector $\underline{c}$ with norm equal to B can be used; otherwise, $\underline{c}$ is unique.) Then $\underline{c} \in S$ and $-\underline{c} \in S$ by earlier arguments. Moreover, $\underline{b} = ((1+\lambda)/2)\cdot\underline{c} + ((1-\lambda)/2)\cdot(-\underline{c})$; therefore, $\underline{b} \in S$ by convexity of S. Now we have proved that the set on the right-hand side of (A5) is a subset of S. If (A5) does not hold, there exists a $\underline{b} \in S$ with $\sum_{k=1}^{K} b_k^2 = A' > A$. Let $\underline{c} = \underline{b}$, and try $\underline{b}$ as a solution to (A3). Then the value of (A3) becomes A', and the optimal value must be at least A'. From (A6) and (A16), we conclude that the optimal value of (A3) for this $\underline{c}$ is $B\sqrt{A'} = \sqrt{AA'} < A'$. This contradiction proves (A5) and Proposition A1.

As pointed out earlier, in the premise of the proposition we only assume that (A3) for any $\underline{c}$ has a solution which satisfies (A4). When (A5) is proved, however, we can conclude that (A4) gives the unique solution to (A3), provided that $\underline{c} \neq \underline{0}$.

The assumption that S is convex, is used only in the last part of the proof. Without this assumption, everything which was said about the function α will still hold. Moreover, we can prove that S contains all K-dimensional vectors with norm equal to B, and no vectors with larger norm. Without convexity, we cannot uniquely characterize S, since we do not know which vectors $\underline{b}$ with $0 < ||\underline{b}|| < B$ belong to S. (We know that $\underline{0} \in S$, since (A4) must hold for $\underline{c} = \underline{0}$.) But these vectors are really irrelevant for the problem, therefore, the convexity assumption is not essential.

Choice of the a-vector

Then we return to the procedure as described in Section 3. As before, we will consider maximization problems of the form (A3); the essential equivalence of (A1) and (A3) is demonstrated above.

A function f is given. In the present version of the model, f represents the "rules of the game," and the purpose of the discussion is to describe f. The self-interested individual solves the following problem, which corresponds to (A3):

(A17) Choose $\underline{a}$ to maximize $\sum_{k=1}^{K} f(a_k)\cdot c_k$, subject to $\sum_{k=1}^{K} |a_k| \leq 1$.

Here c_k is given by (A2); therefore, $\underline{c}$ can be any K-dimensional vector. The constraint in (A17) could have been written $\sum_{k=1}^{K} |a_k| \leq A$ for an arbitrary positive number A. We have restricted ourselves to the case A = 1 in order to simplify notation; nothing of substance would change if we used a fixed but arbitrary $A > 0$.

We want to prove that f must be of the form given in Section 3. If K = 2, this is not true; many other functions satisfy the conditions. This issue is discussed below; here we assume $K \geq 3$.

Suppose that f is given, and define another function f* by interchanging the values of $f(a_0)$ and $f(-a_0)$ for a given $a_0 \neq 0$. That is, f* is given by $f^*(a_0) = f(-a_0)$, $f^*(-a_0) = f(a_0)$, and $f^*(a) = f(a)$ when $|a| \neq |a_0|$. For any $\underline{c}$, (A17) will have essentially the same solution when f is used and when f* is substituted for f. Therefore, we can only hope to prove uniqueness of f if such interchanges are ruled out. To this end, we will assume $f(-a) \leq f(a)$ for all $a > 0$. Note that we do not assume $f(a) > 0$ for $a > 0$ or $f(a) < 0$ for

$a < 0$, but these statements follow from the conclusions obtained below. The assumption does not in any essential way restrict the class of possible functions f; for any f, the statement can be made true by interchanging the values of f(a) and f(-a) for every a which initially violates the assumption. As we have seen, these operations do not change anything.

Proposition A2: Let $K \geq 3$ be an integer, and let f be a real-valued function defined on $[-1,1]$. Suppose that for all $a \in (0,1]$,

$$f(-a) \leq f(a) . \tag{A18}$$

Moreover, assume that for any vector $\underline{c} = (c_1,\ldots,c_K)$, there exists an $\underline{a}^*$ which is a solution to (A17) and which satisfies

$$f(a_k^*) = \alpha c_k \quad , \quad \text{for } k = 1,\ldots,K, \tag{A19}$$

for some number $\alpha > 0$. (The number α may depend on $\underline{c}$.)

Then there exists a number $B > 0$ such that

$$f(a) = \begin{cases} B\sqrt{a} & \text{for} \quad 0 \leq a \leq 1 \\ -B\sqrt{-a} & \text{for} \quad -1 \leq a \leq 0 . \end{cases} \tag{A20}$$

The converse of Proposition A2 can easily be proved: Let f be given by (A20). For any $\underline{c} \neq \underline{0}$, the solution to (A17) is unique and satisfies (A19). For $\underline{c} = \underline{0}$, there exists a solution to (A17) which satisfies (A19), namely $\underline{a}^* = \underline{0}$.

To prove the proposition, we define a set S of K-dimensional vectors by

$$S = \{\underline{b} \mid \text{there exists an } \underline{a} = (a_1,\ldots,a_K) \text{ such that } \sum_{k=1}^{K} |a_k| \leq 1 \text{ and } b_k = f(a_k) \text{ for } k = 1,\ldots,K\}.$$

The premise of Proposition A2 now implies the premise of Proposition A1, except that S need not be convex. Hence we can use everything from the proof of Proposition A1 which does not depend on convexity of S. In particular, for any $\underline{c} \neq \underline{0}$ the number α of (A19) is unique. Hence α can be viewed as a function of $\underline{c}$, and this function is given by (A16), where B is a positive number. We shall prove that (A20) holds for the number B obtained from (A16).

To simplify notation, however, we will assume B = 1. Only trivial modifications are needed in the proof below in order to allow an arbitrary B > 0.

For any $\underline{c} \neq \underline{0}$, if $\underline{a}^*$ is a solution to (A17) which satisfies (A19), we must have

$$\text{(A21)} \qquad f(a_k^*) = \frac{c_k}{\sqrt{c_1^2 + \ldots + c_K^2}} , \quad \text{for } k = 1,\ldots,K .$$

The optimal value of (A17) is

$$\text{(A22)} \qquad \sum_{k=1}^{K} f(a_k^*)c_k = \sqrt{c_1^2 + \ldots + c_K^2} .$$

From now on and until otherwise stated, we shall only consider vectors $\underline{c}$ which satisfy $c_k > 0$ for $k = 1,\ldots,K$. For any such vector there exists, by assumption, a vector $\underline{a}^*$ which solves (A17) and satisfies (A19). There is no loss of generality in assuming $a_k^* \geq 0$ for all k. This follows from (A18) by the following argument: Suppose that $a_k^* < 0$. If $f(a_k^*) < f(-a_k^*)$, the value of (A17) can be increased by substituting the positive number $-a_k^*$ for a_k^*, contradicting the assumption that $\underline{a}^*$ solves (A17). If $f(a_k^*) = f(-a_k^*)$, we can substitute $-a_k^*$ for a_k^* without changing anything of interest; in particular, $f(a_k^*)$ remains unchanged. By (A18), $f(a_k^*) > f(-a_k^*)$ is impossible.

We will assume that all solution vectors $\underline{a}^*$ are chosen so that $a_k^* \geq 0$ for all k. Hence we will only consider the values of f on the interval [0,1]. Moreover, (A21) implies $0 \leq f(a_k^*) \leq 1$ for all possible a_k^*.

We shall construct a function g, defined on (0,1), which is a kind of inverse of f. For any $c \in (0,1)$, let $c_1 = c$ and find positive numbers $c_2,\ldots,c_K$ such that $c_1^2 + \ldots + c_K^2 = 1$. Then find a solution $\underline{a}^*$ to (A17) and (A19), and define

$$g(c) = a_1^* .$$

We have assumed $a_1^* \geq 0$, and the constraint in (A17) requires $|a_1^*| \leq 1$. Hence $0 \leq g(c) \leq 1$. For a given c, there are many possible choices of $c_2,\ldots,c_K$, and for what we know so far, the solution to (A17) and (A19) need not be unique for given $c_1,\ldots,c_K$. We arbitrarily pick one admissible set of values for each c; hence g is well defined on (0,1). From (A21) we get

$$\text{(A23)} \qquad f(g(c)) = f(a_1^*) = \frac{c_1}{\sqrt{c_1^2 + \ldots + c_K^2}} = c .$$

This is the sense in which g is an inverse of f. According to what we have proved so far, we may have $g(f(a)) \neq a$ for some $a \in (0,1)$, in which case g is not really an inverse of f; this possibility will, however, be ruled out later.

Let $c \in (0,1)$ and a be given, and assume $0 \leq a < g(c)$ and $f(a) > c$. If a is substituted for a_1^* in the solution to (A17) which defines g(c), then the constraint of (A17) will still hold, but the criterion will increase since $c_1 > 0$ and $f(a) > c = f(a_1^*)$. This contradicts the assumption that $\underline{a}^*$

is a solution, and for all $c \in (0,1)$ and all a we have

(A24) $$0 \leq a < g(c) \Rightarrow f(a) \leq c \quad .$$

Then let $0 < c < d < 1$. We cannot have $g(c) = g(d)$; that would contradict (A23) and the assumption that f is a single-valued function. Moreover, $g(c) > g(d)$ contradicts (A24). (Substitute $g(d)$ for a, and use (A23).) Hence $g(c) < g(d)$, that is, g is a strictly increasing function on (0,1). Therefore, g can have at most a countable number of points of discontinuity.

Let positive numbers $c_1,\ldots,c_K$ be given and assume $c_1^2 + \ldots + c_K^2 = 1$. Moreover, let an integer k with $1 \leq k \leq K$ be given and suppose that g is continuous at c_k. Finally, let $a_1^*,\ldots,a_K^*$ be a solution to (A17) and (A19). Under these assumptions, we can prove

(A25) $a_k^* = g(c_k)$, and the constraint in (A17) is binding; that is, $a_1^* + \ldots + a_K^* = 1$.

Therefore, in this case at least, the k'th coordinate a_k^* of the solution is unique. To prove the first statement, observe that (A21) implies $f(a_k^*) = c_k$. If $a_k^* < g(c_k)$, continuity of g can be used to find $c < c_k$ such that $g(c) > a_k^*$. This contradicts (A24), with a_k^* substituted for a. If $a_k^* > g(c_k)$, we can find $c > c_k$ such that $g(c) < a_k^*$. Then (A23) gives $f(g(c)) > f(a_k^*)$. If we substitute $g(c)$ for a_k^*, the constraint of (A17) still holds, and the value of the criterion is increased. This contradicts the assumptions that $a_1^*,\ldots,a_K^*$ is a solution and proves that $a_k^* = g(c_k)$. Finally, suppose that $a_1^* + \ldots + a_K^* = 1-\varepsilon$, with $\varepsilon > 0$. There exists a $c > c_k$ such that

$g(c_k) < g(c) < g(c_k) + \varepsilon = a_k^* + \varepsilon$. Again, $g(c)$ can be substituted for a_k^*, to increase the value of (A17) without violating the constraint. This contradiction proves the last part of (A25).

The next step is to prove that g is continuous on (0,1). If this is wrong, find a c with $0 < c < 1$ at which g is not continuous, and define $a' = \lim_{c' \to c^-} g(c')$ and $a'' = \lim_{c' \to c^+} g(c')$. These one-sided limits exist since g is strictly increasing. For the same reason, $a' \leq g(c) \leq a''$, with at least one inequality strict because g is discontinuous at c. Let $\varepsilon = a'' - a'$; then $\varepsilon > 0$. Now we shall choose $c_1, \ldots, c_K$ such that $c_1 = c$, $c_k = c_3$ for $k = 4, \ldots, K$, and $c_1^2 + \ldots + c_K^2 = 1$. This can be done in uncountably many ways. In particular, we can choose c_2 anywhere in the interval $(0, \sqrt{1-c^2})$. For each such value of c_2, there is exactly one value of c_3 which satisfies the conditions, and c_3 chosen this way is a strictly decreasing function of c_2. Since g has at most a countable number of discontinuities, the set of values of c_2 for which g is discontinuous at c_2 or c_3 or both, is countable. Hence there exist possible values of c_2 and c_3 such that g is continuous at both c_2 and c_3. Fix c_2 and c_3 at these values; then we have also fixed $c_k = c_3$ for $k = 4, \ldots, K$. Choose $\delta > 0$ such that $c_2 - \delta < c_2' < c_2 + \delta$ implies $g(c_2) - \varepsilon/2 < g(c_2') < g(c_2) + \varepsilon/2$. We also choose δ so small that $c_2 - \delta > 0$ and $(c_2 + \delta)^2 + c_3^2 + \ldots + c_K^2 < 1$. For each c_2' with $c_2 - \delta < c_2' < c_2$, there exists exactly one $c_1' > c_1$ such that $c_1'^2 + c_2'^2 + c_3^2 + \ldots + c_K^2 = 1$. There are uncountably many possible choices of c_2', and different values of c_2' lead to different c_1'. Hence we can argue in the same way as we did above, to conclude that c_2' can be chosen such that g is continuous at c_1' and c_2'. Now solve the problem (A17) and (A19) for the vector $(c_1', c_2', c_3, \ldots, c_K)$. Equation (A25) applies for all $k = 1,\ldots,K$, and we get $g(c_1') + g(c_2') + g(c_3) + \ldots + g(c_K) = 1$. Similarly, for each c_2'' with $c_2 < c_2'' < c_2+\delta$, there exists a c_1'' with $0 < c_1'' < c_1$ and $c_1''^2 + c_2''^2 + c_3^2 + \ldots + c_K^2 = 1$. We can choose c_2'' such that g is continuous at c_1'' and c_2'', and (A25) gives $g(c_1'') + g(c_2'') + g(c_3) + \ldots + g(c_K) = 1$. Hence $g(c_1') + g(c_2') = g(c_1'') + g(c_2'')$. By the choice of δ, $|g(c_2') - g(c_2'')| < \varepsilon$. Since $c_1'' < c_1 < c_1'$, we have $g(c_1'') \leq a'$ and $g(c_1') \geq a''$, and hence $g(c_1') - g(c_1'') \geq \varepsilon$. This is a contradiction, and we have proved that g is continuous on (0,1). Note that this argument requires $K \geq 3$.

Let $a_0 = \lim_{c \to 0^+} g(c)$; this is well defined and $a_0 \geq 0$. Assume that $a_0 > 0$. Now we choose the positive numbers $c_1,\ldots,c_K$ such that $c_1 = c_2$, $c_k = c_3$ for $k = 4,\ldots,K$, and $c_1^2 + \ldots + c_K^2 = 1$. For any sufficiently small c_3, there exists a c_1 for which this holds. Then (A25) applies and gives $g(c_1) + \ldots + g(c_K) = 2g(c_1) + (K-2)g(c_3) = 1$. We let c_3 tend to 0; then c_1 will tend to $1/\sqrt{2}$. Continuity of g at $1/\sqrt{2}$ and the definition of a_0 give $2g(1/\sqrt{2}) + (K-2)a_0 = 1$. The assumption $a_0 > 0$ now gives $g(1/\sqrt{2}) < 1/2$. Since g is continuous, we can find $c' > 1/\sqrt{2}$ such that $g(c') \leq 1/2$. We define $a = g(c')$, and (A23) gives $f(a) = c' > 1/\sqrt{2}$. Consider the numbers $a_1,\ldots,a_K$, where $a_1 = a_2 = a$ and $a_k = 0$ for $k = 3,\ldots,K$. Then $|a_1| + \ldots + |a_K| \leq 1$, and when the numbers $c_1,\ldots,c_K$ are chosen as described above, the value of (A17) becomes $c_1f(a_1) + \ldots + c_Kf(a_K) = 2c_1c' + (K-2)c_3f(0)$. When c_3 tends to 0 and c_1 tends to $1/\sqrt{2}$, this expression tends to $2(1/\sqrt{2})c' > 1$.

Hence there exists $c_1,\ldots,c_K$ with $c_1^2 + \ldots + c_K^2 = 1$, such that (A17) has a solution in which the value of the criterion exceeds 1. This contradicts (A22). Therefore, $a_0 > 0$ is impossible, and we have $\lim_{c\to 0^+} g(c) = 0$.

For any positive integer M with $2 \leq M \leq K$, we shall prove the following:

(A26) Let $c_1,\ldots,c_M$ be positive numbers such that $c_1^2 + \ldots + c_M^2 = 1$. Then $g(c_1) + \ldots + g(c_M) = 1$.

To prove this, choose numbers $c_1^{(j)},\ldots,c_K^{(j)}$ for all positive j, such that:

(a) For all j and all $k = 1,\ldots,K$, $c_k^{(j)} > 0$.

(b) For all j, $(c_1^{(j)})^2 + \ldots + (c_K^{(j)})^2 = 1$.

(c) For $k = 1,\ldots,M$, $\lim_{j\to\infty} c_k^{(j)} = c_k$.

(d) For $k = M+1,\ldots,K$, $\lim_{j\to\infty} c_k^{(j)} = 0$.

For any j, (A25) implies that $g(c_1^{(j)}) + \ldots + g(c_K^{(j)}) = 1$.

Then (A26) follows from continuity of g and the fact that $\lim_{c\to 0^+} g(c) = 0$.

We can let M = 2 in (A26). By letting c_1 tend to 1 and c_2 tend to 0, in such a way that $c_1^2 + c_2^2 = 1$, we get $\lim_{c\to 1^-} g(c) + \lim_{c\to 0^+} g(c) = 1$. Hence $\lim_{c\to 1^-} g(c) = 1$. Since g is also continuous and strictly increasing, we can now conclude that for each $a \in (0,1)$ there exists a unique $c \in (0,1)$ such that $g(c) = a$. Then (A23) gives $f(a) = c$. Therefore, f on (0,1) and g on (0,1) are inverse functions. From what we have proved about g, we can then conclude that f is continuous and strictly increasing. Moreover, $0 < f(a) < 1$ for $a \in (0,1)$.

An inverse of (A26) can now be established. For any positive integer M with $2 \le M \le K$, we have:

(A27) Let $a_1, \ldots, a_M$ be positive numbers such that $a_1 + \ldots + a_M = 1$. Then $(f(a_1))^2 + \ldots + (f(a_M))^2 = 1$.

To prove this, let $a_1, \ldots, a_M$ be given, and define, for $k = 1, \ldots, M$,

(A28)
$$c_k = \frac{f(a_k)}{\sqrt{(f(a_1))^2 + \ldots + (f(a_M))^2}} .$$

Then $c_k > 0$ and $c_1^2 + \ldots + c_M^2 = 1$, and (A26) implies $g(c_1) + \ldots + g(c_M) = 1$. If the denominator in (A28) is less than 1, $c_k > f(a_k)$ for $k = 1, \ldots, M$, which implies $g(c_k) > g(f(a_k)) = a_k$ and contradicts the assumption $a_1 + \ldots + a_M = 1$. If the denominator is greater than 1, a similar contradiction follows. Hence we must have $\sqrt{(f(a_1))^2 + \ldots + (f(a_M))^2} = 1$, which gives the conclusion of (A27).

By applying (A27) with $M = 2$ and $a_1 = a_2 = 1/2$, we get $f(1/2) = 1/\sqrt{2}$. More generally, we shall prove that $f(a) = \sqrt{a}$ whenever $a = i/2^j$ for some positive integers i and j with $i < 2^j$. The proof is by induction on j, and the case $j = 1$ has already been covered. Assume that the statement holds for a given j and all possible i, and let $a = i/2^{j+1}$ for integer i. First suppose $0 < a < 1/2$, that is, $0 < i < 2^j$. Define $i' = 2^j - i$ and $a' = i'/2^j$. Then $a + a + a' = 1$, and (A27) implies

$(f(a))^2 + (f(a))^2 + (f(a'))^2 = 1$. The induction hypothesis gives $f(a') = \sqrt{a'}$, hence this implies $f(a) = \sqrt{a}$. If $a = i/2^{j+1}$ satisfies $1/2 < a < 1$, we have just shown that $f(1-a) = \sqrt{1-a}$. An application of (A27) with $M = 2$ proves that $f(a) = \sqrt{a}$. The only remaining case is $a = 1/2$, which has been taken care of above. This concludes the induction step. Note that (A27) is used only for $M = 2$ and $M = 3$; hence the proof works for any $K \geq 3$.

The square-root function and the function f are both continuous on (0,1). They are equal on all points of the form $i/2^j$, which form a dense subset of (0,1). Hence (A20) holds for $a \in (0,1)$.

If $f(0) > 0$, we can choose c such that $0 < c < f(0)$ and let $a = 0$. Then $g(c) > a$, and we have a contradiction to (A24). Hence $f(0) \leq 0$.

We now drop the assumption that $\underline{c}$ satisfies $c_k > 0$ for all k. Choose any number c_1 with $-1 < c_1 < 0$, and let $c_2,\ldots,c_K$ be positive numbers such that $c_1^2 + \ldots + c_K^2 = 1$. Find a solution $\underline{a}^*$ to (A17) which satisfies (A19). By an argument used above, (A18) can be used to prove that there is no loss of generality in assuming $a_1^* \leq 0$ and $a_k^* \geq 0$ for $k = 2,\ldots,K$. Clearly we have:

(A29) $$|a_1^*| + \ldots + |a_k^*| \leq 1 \quad .$$

Moreover, (A21) gives:

(A30) $$f(a_k^*) = c_k \quad , \quad \text{for } k = 1,\ldots,K \; .$$

Since $f(0) \leq 0$, we must then have $a_k^* > 0$ for $k = 2,\ldots,K$. Then (A29) implies that no a_k^* has absolute value 1; therefore, $0 < a_k^* < 1$ for $k \geq 2$. We know the function f on (0,1), and (A30) implies $a_k^* = c_k^2$ for $k \geq 2$. The constraint (A29) must be binding; otherwise, the value of (A17) could be increased by increasing a_2^*. Therefore, $|a_1^*| = 1 - a_2^* - \ldots - a_K^* = 1 - c_2^2 - \ldots - c_K^2 = c_1^2$. We have assumed $a_1^* \leq 0$; hence (A30) gives $f(a_1^*) = c_1 = -\sqrt{-a_1^*}$. As we vary c_1 in (-1,0), a_1^* takes on all values in (-1,0). Thus we have proved (A20) for $a \varepsilon (-1,0)$.

We have determined f everywhere except on -1, 0 and 1. By choosing $\underline{c} = (1,0,\ldots,0)$ and $\underline{c} = (-1,0,\ldots,0)$, we see from (A21) that -1, 0 and 1 must occur as values of f. Since $0 < |a| < 1$ implies $0 < |f(a)| < 1$, we can conclude that f(-1), f(0) and f(1) must be equal to -1, 0 and 1, not necessarily in that order. Assume $f(0) = -1$, let $c = (-1,\ldots,-1)$, and try $\underline{a} = \underline{0}$ as a solution to (A17). The constraint is satisfied, and the value of (A17) is K. This contradicts (A22), which implies that the optimal value is $\sqrt{K}$. We have earlier proved $f(0) \leq 0$; hence $f(0) = 0$. Now f(-1) and f(1) must be equal to -1 and 1, and (A18) implies $f(-1) = -1$ and $f(1) = 1$. Then (A20) is proved for all $a \varepsilon [-1,1]$, and the proof of Proposition A2 is complete.

<u>The Case K = 2</u>

If K = 2, Proposition A2 does not hold. An example of a function which satisfies the premise but not the conclusion is

$$\text{(A31)} \qquad f(a) = \sin(\tfrac{\pi}{2} a) \quad , \quad \text{for} \quad a \,\varepsilon\, [-1,1] \; .$$

This function satisfies (A18). It is also continuous and strictly increasing.

To find the solution to (A17), let us first assume $c_1 > 0$ and $c_2 > 0$. The solution $\underline{a}^*$ will clearly satisfy $a_1^* \geq 0$ and $a_2^* \geq 0$. On [0,1], the function f is strictly increasing, differentiable and strictly concave. Then the solution must satisfy $a_1^* + a_2^* = 1$. Since $f'(1) = 0$, we can rule out corner solutions, that is, solutions with $a_1^* = 1$ or $a_2^* = 1$. Hence we can use the first-order conditions, which give

$$c_1 f'(a_1^*) = c_2 f'(a_2^*) \quad .$$

We have $f'(a) = \frac{\pi}{2}\cos(\frac{\pi}{2}a) = \frac{\pi}{2}\sin(\frac{\pi}{2} - \frac{\pi}{2}a) = \frac{\pi}{2}f(1-a)$; hence this implies

$$c_1 \frac{\pi}{2} f(a_2^*) = c_2 \frac{\pi}{2} f(a_1^*) \quad ,$$

from which (A19) follows directly. A straightforward computation gives

(A32) $$a_k^* = \frac{2}{\pi}\arcsin\left(\frac{c_k}{\sqrt{c_1^2 + c_2^2}}\right)$$

for k = 1,2. It is easy to see that (A32) holds for all $\underline{c} \neq \underline{0}$. For $\underline{c} = \underline{0}$, $\underline{a}^* = \underline{0}$ is a solution to (A17). In any case, (A19) holds, and the premise of Proposition A2 is established.

For $\underline{c} \neq \underline{0}$, the number α of (A19) will be of the form (A16). The set S, defined in the beginning of the proof of Proposition A2, will satisfy (A5); this corresponds to the fact that Proposition A1 is true for K = 2. But f is obviously not of the form given by (A20).

In addition to the increasing and continuously differentiable example given by (A31), we can find discontinuous or non-monotone functions which satisfy the premise of Proposition A2 when K = 2. For example, let γ be any number with $0 < \gamma \leq 1$, and define f by:

$$\text{(A33)}\quad \begin{aligned} f(a) &= \sin(\tfrac{\pi}{2\gamma}\, a) && \text{for} \quad 0 \le a \le \tfrac{\gamma}{2} \\ f(a) &\in [0, \tfrac{1}{\sqrt{2}}) && \text{for} \quad \tfrac{\gamma}{2} < a < 1 - \tfrac{\gamma}{2} \\ f(a) &= \sin(\tfrac{\pi}{2\gamma}\, a - \tfrac{\pi}{2\gamma} + \tfrac{\pi}{2}) && \text{for} \quad 1 - \tfrac{\gamma}{2} \le a \le 1 \\ f(a) &= -f(-a) && \text{for} \quad -1 \le a \le 0\ . \end{aligned}$$

If $\underline{a}^*$ solves (A17) for some $\underline{c} \neq \underline{0}$, a_1^* and a_2^* cannot lie in the intervals $(-1 + \gamma/2, -\gamma/2)$ or $(\gamma/2, 1 - \gamma/2)$. It is not difficult to prove (A19).

If we set $\gamma = 1$ in (A33), we are back in (A31). For $\gamma < 1$, we have considerable freedom in defining f on $(\gamma/2, 1 - \gamma/2)$; hence functions f with various properties can be constructed.

If we look at the proof of Proposition A2, the function g can be defined and will be strictly increasing even if K = 2, but we are not able to prove that it is continuous. (In (A33), with $\gamma < 1$, g will be discontinuous at $1/\sqrt{2}$, whether or not f is continuous.) Even if g is continuous, the rest of the proof breaks down, since (A26) and (A27) can only be established for M = 2. In the heuristic proof in Section 3, equation (14) only allows us to conclude

$$f(a_1^*)f'(a_1^*) = f(a_2^*)f'(a_2^*)\ ,$$

when (a_1^*, a_2^*) is a solution to (A17) for some (c_1, c_2). (Here we have in mind the case where a_1^* and a_2^* are both positive; similar remarks hold for the other cases.) If the constraint in (A17) is always binding, this implies:

$$f(a)f'(a) = f(1-a)f'(1-a) \;, \qquad \text{(A34)}$$

for all a such that (a, 1-a) can occur as a solution. When f is given by (A31), (A34) holds for all $a \in (0,1)$. For (A33), the formula holds only if $0 < a < \gamma/2$ or $1 - \gamma/2 < a < 1$. In any case, (A34) does not suffice to conclude that $f(a)f'(a)$ is a constant function.

APPENDIX B

In this appendix, we report in detail on the computational experiments mentioned in Section 5.

The Preferences

In the experiments, we use a class of preferences which is a special case of the preferences described in the paper.

Preferences for alternative public goods vectors are derived from a fixed tax system and from preferences over private and public goods. Such derived preferences are discussed in Section 2; see also references to Zeckhauser and Weinstein (1974). The public goods are measured in terms of expenditures. That is, if the bundle $\underline{x} = (x_1,\ldots,x_K)$ is chosen, the amount of money spent on the k'th public good is x_k, where the currency unit is fixed but arbitrary. The total costs of the public goods are divided among the participants in fixed shares. We let θ_i denote the share paid by individual i. These shares must satisfy $\sum_{i=1}^{N} \theta_i = 1$.

The initial wealth of individual i is denoted $\tilde{R}_i$. If public goods bundle $\underline{x}$ is chosen, the amount of money i can spend on private goods is

$$\text{(B1)} \qquad \tilde{x}_{i0} = \tilde{R}_i - \theta_i \sum_{k=1}^{K} x_k \quad .$$

We assume that each individual has preferences over private and public goods which can be represented by a "constant elasticity of substitution" utility function. Then the induced valuation function for public goods is

$$\text{(B2)} \qquad v_i(\underline{x}) = \left(-\frac{1}{\rho_i}\right)\left(\tilde{\alpha}_{i0}\tilde{x}_{i0}^{-\rho_i} + \tilde{\alpha}_{i1}x_1^{-\rho_i} + \ldots + \tilde{\alpha}_{iK}x_K^{-\rho_i}\right) ,$$

where $\tilde{x}_{i0}$ is given by (B1). Here ρ_i is a parameter which represents the elasticity of substitution; we must have $\rho_i > -1$ and $\rho_i \neq 0$. The parameters

$\tilde{\alpha}_{ik}$ for k = 0,1,...,K must be positive; they represent the weights person i puts on private goods and on the various public goods. To be precise, the function given by (B1) is a strictly increasing transformation of a function with constant elasticity of substitution; the transformation is chosen so as to make v_i strictly concave. When ρ_i tends to 0, v_i essentially converges to the Cobb-Douglas utility function, given by

$$\text{(B3)} \qquad v_i(\underline{x}) = \tilde{\alpha}_{i0}\, \ell n\, \tilde{x}_{i0} + \tilde{\alpha}_{i1}\, \ell n\, x_1 + \ldots + \tilde{\alpha}_{iK}\, \ell n\, x_K \quad .$$

The derivative of v_i with respect to the k'th variable, evaluated at $\underline{x}$, is given by

$$\text{(B4)} \qquad c_{ik} = -\theta_i \frac{\tilde{\alpha}_{i0}}{\tilde{x}_{i0}^{\rho_i+1}} + \frac{\tilde{\alpha}_{ik}}{x_k^{\rho_i+1}} \quad .$$

This formula applies both for $\rho_i \neq 0$ and $\rho_i = 0$. It captures all interesting aspects of v_i, and we can forget (B2) and (B3).

For convenience, we shall make some changes in the parameters. We define, for i = 1,...,N:

$$\sigma_i = \frac{1}{\rho_i+1}$$

$$R_i = \frac{\tilde{R}_i}{\theta_i}$$

$$x_{i0} = \frac{\tilde{x}_{i0}}{\theta_i} = R_i - \sum_{k=1}^{K} x_k$$

$$\alpha_{i0} = \tilde{\alpha}_{i0}^{\sigma_i}\ \theta_i^{\sigma_i-1}$$

$$\alpha_{ik} = \tilde{\alpha}_{ik}^{\sigma_i}\ , \quad \text{for } k = 1,\ldots,K.$$

Then (B4) becomes:

(B5) $$c_{ik} = -\left[\frac{\alpha_{i0}}{x_{i0}}\right]^{1/\sigma_i} + \left[\frac{\alpha_{ik}}{x_k}\right]^{1/\sigma_i} .$$

In all the examples, we let R_i be the same for every i. This is equivalent to assuming that the tax shares $\theta_1,\ldots,\theta_N$ are proportional to the individuals' initial wealths $\tilde{R}_1,\ldots,\tilde{R}_N$. The assumption implies that x_{i0} is also equal for all i. We shall write R and x_0 instead of R_i and x_{i0}. Moreover, we let $R = 10(K+1)$; this is not a substantive assumption, but simply the choice of a currency unit.

Under these assumptions, R can be interpreted as the total resources of society. Moreover, x_0 and $x_1,\ldots,x_K$ represent the resources spent on private goods and on the various public goods. Individual i's net endowment of private goods will be $\theta_i x_0$.

The optimal decision, from person i's point of view, is easily seen to be given by:

(B6) $$x_k^{(i)} = \alpha_{ik} \frac{R}{\sum_{m=0}^{K} \alpha_{im}} , \quad \text{for } k = 0,\ldots,K .$$

In the examples used below, the numbers $\alpha_{i0},\alpha_{i1},\ldots,\alpha_{iK}$ will on the average be of the same order of magnitude. This is perhaps not realistic; it implies that the typical individual prefers that only a fraction $1/(K+1)$ of society's resources be spent on private goods. But there is no reason to believe that the computational experiments would give different results if we used preferences in which more weight were placed on private goods.

The Procedure

The distribution of influence can be described by the numbers $A_1,\ldots,A_N$ or $B_1,\ldots,B_N$, where $B_i = \sqrt{A_i}$. We define

(B7) $$A_0 = \max_{1 \le i \le N} A_i ,$$

and similarly

(B8) $$B_0 = \max_{1 \le i \le N} B_i .$$

When a tentative decision $\underline{x}$ is announced, person i will react in the following way, provided that $\underline{x}$ is not i's optimal point: The derivatives of v_i at $\underline{x}$ are computed by (B5). This gives a vector $\underline{c}_{i.} = (c_{i1},\ldots,c_{iK})$. Then i's response $\underline{b}_{i.} = (b_{i1},\ldots,b_{iK})$ is computed as follows:

(B9) $$b_{ik} = \frac{B_i}{\|\underline{c}_{i.}\|} c_{ik} , \quad \text{for } k = 1,\ldots,K .$$

Hence $\underline{b}_{i.}$ is a vector which is parallel to $\underline{c}_{i.}$ and has length B_i, as required by the arguments in Sections 3 and 4. (Note that the function f and the allocation $a_{i1},\ldots,a_{iK}$ of i's influence points do not appear here; in the computational examples we can go directly to the votes for movement, given by $\underline{b}_{i.}$.)

Two additional parameters, denoted η and ε, play a role in the procedure. The parameter η is used to decide when the step length shall be reduced, while ε represents the required degree of accuracy; the exact interpretation of η and ε is given below.

Now we are ready to present the algorithm. The description of the algorithm is divided into five parts:

Part 1. For each i, let $\underline{x}^{(i)} = (x_1^{(i)},\ldots,x_K^{(i)})$ be i's optimal point, as given by (B6). This is announced as a tentative decision, and responses are received from all individuals but i. With the responses given by (B9), we compute

$$\underline{d}^{(i)} = \sum_{j\neq i} \underline{b}_{j\cdot} \quad .$$

The point $\underline{x}^{(i)}$ is an equilibrium if and only if $\|\underline{d}^{(i)}\| \leq B_i$.

If $\underline{x}^{(i)}$ is an equilibrium for at least one i, we stop. Otherwise, we go on to Parts 2 - 5.

Part 1 is included in order to avoid asking person i to react when $\underline{x}^{(i)}$ is the tentative decision. (This reaction would not be well defined, since the denominator $\|\underline{c}_{i\cdot}\|$ in (B9) is 0.) We have assumed that all $\underline{x}^{(i)}$ are different, but the description can easily be modified so as to take account of the possibility that several of these points are identical. If it is concluded that no $\underline{x}^{(i)}$ is an equilibrium, we can safely assume that the iteration process of Parts 2 - 5 will never ask for responses which are not well defined.

Part 2. As a starting point for the iteration process, we choose

(B10) $$\underline{x} = \frac{\sum_{i=1}^{N} A_i \underline{x}^{(i)}}{\sum_{i=1}^{N} A_i} \quad .$$

Part 3. All individuals are asked to react to the given $\underline{x}$. From the responses, as given by (B9), we compute:

(B11) $$\underline{d} = \sum_{i=1}^{N} \underline{b}_{i\cdot} \quad .$$

Then we let $\underline{x}' = \underline{x} + \underline{d}$ and ask people to react to $\underline{x}'$ as a tentative decision. Next $\underline{d}'$ is computed from the responses, in analogy with (B10). Finally, we let $\underline{x}'' = \underline{x}' + \underline{d}'$.

Part 4. If $\|d\| \leq B_0\varepsilon$ and $\|\underline{d}'\| \leq B_0\varepsilon$, we stop and declare $\underline{x}''$ to be a sufficiently good approximation to the equilibrium. Otherwise, we go on to Part 5.

Note that the stopping criterion depends on B_0, the maximum of $B_1,\ldots,B_N$. The iteration process will never stop simply because the numbers $B_1,\ldots,B_N$ (or $A_1,\ldots,A_N$) are small. Of course, if these numbers are small, the norms of $\underline{d}$ and $\underline{d}'$ will also be small. When the procedure stops by the criterion used here, we know that people's responses actually cancel each other out, and that this has happened for two consecutive steps in the iteration.

Part 5. Consider the condition

$$\text{(B12)} \qquad \frac{\underline{d} \cdot \underline{d}'}{\|\underline{d}\| \cdot \|\underline{d}'\|} \geq \eta \quad .$$

Here η is a parameter, which is supposed to have a value slightly less than 1. The left-hand side of (B12) is the cosine of the angle between $\underline{d}$ and $\underline{d}'$.

If (B12) is satisfied, the increments $\underline{d}$ and $\underline{d}'$ point in approximately the same direction. Then we substitute $\underline{x}''$ for $\underline{x}$ and return to Part 3 for a new round in the iteration.

If (B12) is not satisfied, there is reason to believe that we have "overshot" the equilibrium, or that we may soon do so. Then we replace the number B_i, for $i = 1,\ldots,N$, by $B_i/2$. (This corresponds to replacing A_i by $A_i/4$.) Note that this also affects the value of B_0, see equation (B8). Essentially, we have cut the step length in half. We return to Part 3 for a new round in the iteration, with $\underline{x}$ unchanged. (Hence we discard the vectors $\underline{d}$, $\underline{d}'$, $\underline{x}'$ and $\underline{x}''$, which were computed a moment ago in Part 3.)

The Parameters

In some of the examples we have used, the parameters were randomly generated. Here we describe the probability distribution from which the values were drawn.

The randomly chosen parameters are A_i and σ_i for $i = 1,\ldots,N$, and α_{ik} for $i = 1,\ldots,N$, $k = 0,\ldots,K$. All drawings are independent. (Because of rescaling, the numbers A_i will not, however, be stochastically independent.)

To choose $A_1,\ldots,A_N$, we let $A_1^*,\ldots,A_N^*$ be N independent drawings from a uniform distribution on [0,1]. Then we compute $A_0^* = \max_{1\leq i\leq N} A_i^*$ and set $A_i = A_i^*/A_0^*$ for $i = 1,\ldots,N$. Hence we always get $A_0 = 1$.

Concerning σ_i, the probability shall be 0.5 for each of $\sigma_i < 1$ and $\sigma_i > 1$. If $\sigma_i < 1$, then σ_i is uniformly distributed on [0.1, 1]. If $\sigma_i > 1$, then $1/\sigma_i$ is uniformly distributed on [0.1, 1].

Each α_{ik} is uniformly distributed on [0, 1].

The Results

We have applied the procedure to a number of typical cases, and to cases generated randomly, as described above. In all cases, we use $\varepsilon = 0.001$ and $\eta = 0.9$.

The interesting result is the number of steps required to reach an equilibrium. The "number of steps" given in the tables below, is the number of times Part 3 is performed. Part 1 is not counted; therefore, no "steps" are needed if some $\underline{x}^{(i)}$ is an equilibrium. Each step requires two rounds of responses from the individuals. A step is counted even if (B12) fails, so that the step does not bring the procedure forwards; responses must be obtained in any case.

The value $\varepsilon = 0.001$ represents a very high degree of accuracy. The number of steps needed must be viewed in light of this.

In the tables, we do not give the numbers α_{ik}, but instead the point $\underline{x}^{(i)}$. Note that R is always equal to 10(K+1). When R is given, $\underline{x}^{(i)}$ contains all the information in $\alpha_{i0},\ldots,\alpha_{iK}$, except an unimportant factor of scale.

We first present some "typical" cases. The optimal points are the same in all these cases; we only vary A_i and σ_i. Note that the second and the fifth examples are equal except for the scale of the numbers A_i; this affects the numbers of steps needed.

In the randomly generated cases, the average number of steps needed is 17. In more than half of the cases, a vector $\underline{x}^{(i)}$ is an equilibrium and no step is needed. If these cases are excluded, the average number of steps is 37.

Some "Typical" Cases

K = 2, N = 3

i	$\underline{x}^{(i)}$		A_i	σ_i	Equilibrium/number of steps
1	5	10	5	0.5	11.51 8.33
2	10	15	5	0.5	15
3	15	5	5	0.5	

i	$\underline{x}^{(i)}$		A_i	σ_i	Equilibrium/number of steps
1	5	10	5	1	11.40 8.27
2	10	15	5	1	8
3	15	5	5	1	
1	5	10	5	2	11.43 8.17
2	10	15	5	2	9
3	15	5	5	2	
1	5	10	5	0.5	11.40 8.01
2	10	15	5	1	11
3	15	5	5	2	
1	5	10	1	1	11.40 8.28
2	10	15	1	1	18
3	15	5	1	1	
1	5	10	0.3	1	11.76 9.14
2	10	15	1	1	31
3	15	5	0.6	1	

Randomly Generated Cases

K = 2, N = 3

i	$\underline{x}^{(i)}$		A_i	σ_i	Equilibrium/number of steps
1	11.94	9.70	0.43	3.63	16.99 7.77
2	18.41	6.06	0.40	1.22	18
3	17.28	10.79	1	1.42	
1	18.89	1.75	0.55	1.01	$\underline{x}^{(3)}$
2	6.56	10.87	0.74	0.77	0
3	11.62	7.88	1	1.18	
1	10.18	8.51	0.04	1.41	$\underline{x}^{(2)}$
2	5.90	14.92	1	0.39	0
3	2.18	5.27	0.56	0.51	
1	10.36	6.06	0.97	1.33	$\underline{x}^{(2)}$
2	8.91	8.03	1	9.03	0
3	4.17	13.53	0.57	4.75	
1	9.69	6.74	1	0.26	$\underline{x}^{(1)}$
2	3.57	12.26	0.37	1.28	0
3	14.03	3.55	0.71	0.45	
1	3.35	9.92	0.82	0.46	11.34 6.86
2	12.61	1.88	0.13	2.95	53
3	24.42	4.74	1	1.11	

i	$\underline{x}^{(i)}$		A_i	σ_i	Equilibrium/number of steps	
1	3.85	13.57	0.65	2.83	$\underline{x}^{(2)}$	
2	7.98	9.59	0.86	1.36	0	
3	8.84	9.51	1	1.12		
1	6.89	4.23	0.44	2.11	$\underline{x}^{(3)}$	
2	14.82	4.57	0.08	1.82	0	
3	11.50	11.57	1	1.25		
1	4.55	13.30	0.83	0.99	11.69	8.79
2	12.82	13.70	0.62	1.40	20	
3	19.30	0.55	1	1.17		
1	11.51	17.46	0.67	0.46	$\underline{x}^{(3)}$	
2	2.24	7.86	1	2.48	0	
3	7.52	13.61	0.63	0.68		
1	10.14	5.17	0.87	1.17	4.86	15.59
2	2.16	17.95	1	0.21	26	
3	5.89	16.61	0.33	0.87		
1	10.21	10.27	0.51	3.41	10.88	9.90
2	13.94	7.38	1	8.08	64	
3	9.37	12.18	0.32	0.83		
1	16.31	3.38	0.73	0.23	$\underline{x}^{(2)}$	
2	8.38	11.40	0.90	0.66	0	
3	7.30	10.40	1	0.98		

K = 2, N = 4

i	$\underline{x}^{(i)}$		A_i	σ_i	Equilibrium/number of steps
1	17.40	8.36	0.38	2.37	$\underline{x}^{(4)}$
2	10.18	9.96	0.32	0.66	0
3	7.97	3.75	0.71	0.38	
4	10.40	7.82	1	0.30	

i	$\underline{x}^{(i)}$		A_i	σ_i	Equilibrium/number of steps	
1	3.72	11.37	0.65	0.86	10.33	14.26
2	12.33	11.07	0.81	2.90	94	
3	13.16	15.57	0.90	0.23		
4	9.61	15.24	1	0.82		

i	$\underline{x}^{(i)}$		A_i	σ_i	Equilibrium/number of steps
1	27.37	0.07	1	1.23	$\underline{x}^{(4)}$
2	12.36	16.89	0.37	0.87	0
3	3.22	5.89	0.43	0.56	
4	10.44	3.89	0.93	1.64	

K = 2, N = 5

i	$\underline{x}^{(i)}$		A_i	σ_i	Equilibrium/number of steps	
1	16.21	11.71	0.97	3.22	11.08	9.42
2	10.57	5.50	1	0.79	20	
3	9.95	9.87	0.33	0.43		
4	10.45	8.45	0.23	0.96		
5	16.03	13.84	0.86	1.86		

i	$\underline{x}^{(i)}$		A_i	σ_i	Equilibrium/number of steps
1	13.65	15.14	0.62	6.86	$\underline{x}^{(3)}$
2	12.48	9.29	0.10	1.41	0
3	12.80	12.07	1	0.48	
4	4.80	3.20	0.25	0.41	
5	7.91	10.40	0.60	0.44	

i	$\underline{x}^{(i)}$		A_i	σ_i	Equilibrium/number of steps
1	5.74	9.13	0.55	1.70	$\underline{x}^{(2)}$
2	11.64	10.94	0.74	0.81	0
3	11.20	13.66	1	1.50	
4	16.48	2.72	0.82	0.91	
5	6.87	11.82	0.08	2.19	

1	10.90	8.49	0.27	0.17	11.72	4.23
2	13.29	4.82	0.59	1.48	22	
3	13.09	3.26	1	1.09		
4	5.79	7.80	0.24	1.64		
5	0.11	3.43	0.85	0.61		

1	24.39	3.56	1.00	0.71	12.53	9.27
2	2.71	10.46	0.81	0.47	34	
3	11.35	10.07	0.68	0.71		
4	14.27	9.78	1	1.89		
5	10.40	4.59	0.22	2.63		

K = 4, N = 5

i	$\underline{x}^{(i)}$				A_i	σ_i	Equilibrium/number of steps			
1	7.82	15.47	13.94	11.11	0.56	1.37	8.07	12.92	4.80	10.27
2	6.69	11.92	4.70	14.61	0.07	0.55	23			
3	6.44	17.21	6.90	5.16	0.05	3.92				
4	4.70	15.13	4.38	10.71	0.75	1.27				
5	21.44	2.36	1.32	7.06	1	0.98				