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Proportional Representation in Variable-Size Legislatures

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Abstract. This paper examines a method to achieve proportional representation (PR) for parties in a legislature that allows the size of the legislature to vary as a function of voting results. The legislature's *base* consists of one elected candidate from each voting district. Seats are then added to underrepresented parties to approximate their nationwide vote proportions. Additions are governed by rules designed to honor the single-member district concept, to keep the increase manageable, and to satisfy other desirable criteria.

These rules work well in the two-party case although they limit the extent to which true PR is achieved. When there are three or more parties, it is necessary to relax the rules if reasonable moves toward PR are to occur.

1. Introduction

> *A scheme of proportional representation attempts to secure an assembly whose membership will, so far as possible, be proportionate to the volume of the different shades of political opinion held throughout the country; the microcosm is to be a true reflexion of the macrocosm.* [2, p. 75]

In modern democracies, Black's "shades" are usually taken to mean political parties. A variety of ways have been devised to approximate their proportional representation (PR) in legislatures, some of which will be mentioned shortly, but our focus will not be on comparative analyses of PR systems.

Our main purpose is rather to formulate and analyze a particular method of apportioning seats in a legislature that is unorthodox but has a number of attractive features, including some accommodation to PR. The unorthodox feature is a variable-size legislature. That is, while the number of election districts remains fixed, the size of the legislature can vary from election to election by perhaps as much as 20%–30% or more.

The only legislature we know of that allows such variation today is the bicameral legislature of Puerto Rico. There, votes for governor determine relative proportions of seats for the parties in the two houses. No adjustment is made unless the largest party in a house wins more than two-thirds of the seats in district elections. When this happens, that house can be increased by as much as one-third to ameliorate underrepresentation of minority parties. The full increase applies only if the largest party obtains fewer than two-thirds of the votes for governor (but more than two-thirds of the seats); otherwise, additions (of less than one-third) are made to the extent needed to achieve PR [9].

Other schemes for PR add a fixed number of seats to those won in single-member or multi-member election districts. Denmark and Sweden use total votes, summed over each party's district candidates, as the basis for allocating additional seats. In elections to West Germany's Bundestag (496 seats) and Iceland's Parliament (60 seats), voters vote twice, once for district representatives and once for a party. Half of the Bundestag is chosen from party lists, on the basis of the party vote, to achieve approximate PR. Similar systems have been proposed in Canada [7, 8] and Britain [3, 6].

To keep matters relatively simple, we shall assume that there is a unicameral parliament and refer to this as the legislature. The legislature's *base* consists of one elected representative from each of m districts of roughly equal populations. Within-district elections can be conducted by any practicable method, such as plurality voting – vote for exactly one candidate – or approval voting – vote for as many candidates as one likes or finds acceptable [4]. Each candidate is a member of a political party, but voters, in addition to voting for candidates, indicate their favored party in a plurality election. Thus, as in other schemes mentioned earlier, each voter votes twice, once for candidate(s) and once for a single party. The two votes are distinguished, for some voters may prefer a local candidate who does not belong to the party they favor nationally.

The total votes for the parties are used to adjust the legislature's size by adding defeated candidates from parties that are underrepresented in the legislature's base. For example, a party that gets 20% of the party vote but has only a 10% representation in the base may get several extra seats, which may, for example, go to its defeated candidates who had the largest district vote totals or percentages. (This is the method used to allocate additional seats in Puerto Rico.) In making such additions, we consider several general principles, including the following three:

1. Only underrepresented parties can achieve additional seats.

2. The additions to these parties cannot increase their proportions in the augmented legislature beyond their proportions of the party vote, nor can they decrease the proportions of initially overrepresented parties in the legislature below *their* proportions of the party vote.

3. Other things being equal, a party cannot achieve a greater proportion in the augmented legislature by winning fewer districts.

These and other principles are discussed further in Sect. 3, following the formulation of a general model in the next section that highlights problems of achieving PR. In later sections, we show how these principles affect the number of additions when one's purpose is to achieve PR as nearly as is practicable.

Because it is easiest to analyze, the two-party case will be examined in detail in Sect. 4. In Sect. 5, we discuss conflicts between the principles and approximating PR, show how the principles mentioned above affect additions when there is one large party and two or more smaller parties, and suggest relaxations of our rules that more realistically accommodate the goal of achieving PR.

The most significant effect of our principles – even in the two-party case – is to limit additions that would allow PR to be realized in the augmented legislature when party representation, based on district elections, is disproportionate to the national party vote (i.e., when the base is unbalanced). For example, if there are 100 districts and two parties, and if one of the parties has 49% of the party vote but wins in only 10 of the 100 districts, then the most additional seats it can receive is 39 if the principles are to be met. It would then have 49 of the 139 seats in the augmented legislature, for a representation of about 35%. Thus, our method would increase its representation from 10% to 35%, which is considerably below its percentage of the party vote.

The main case we examine in Sect. 5 assumes that there is one large party and several smaller parties – none of which wins as many districts as its proportional share of the party vote. Our results for this case show that, under the foregoing principles, the smaller parties can receive additional seats up to the number they would have won if they had met their share of the party vote. For example, if there are 100 districts and five parties that win (3, 10, 17, 28, 42) districts and have (5.1, 12.1, 20.2, 29.3, 33.3)% of the party vote, then the maximum additions to the smaller parties give an augmented legislature of (5, 12, 20, 29, 42) for a total of 108 seats. The percentage proportions of the parties in the augmented legislature are then about (4.6, 11.1, 18.5, 26.9, 38.9).

Although the augmented seat proportions for the smaller parties come close to their vote proportions in this example, there is a disturbing paradoxical result. The fourth party simultaneously gained an extra seat and slipped below its base proportion of 0.28. Thus, not only may satisfying the principles leave the augmented legislature well short of PR, but their satisfaction may also create violations of other principles deemed desirable – in this case, that an initially underrepresented party should not do worse, proportionately, in the augmented legislature. We shall say more about this principle in Sects. 3 and 5.

2. General Formulation and Goals

We let m denote the number of single-member election districts and n the number of political parties, indexed by $i = 1, \ldots, n$. The data from the district elections are summarized by

s_i = number of districts won by party i,
p_i = proportion of the national party vote for party i,

with $s_1 + \ldots + s_n = m$ and $p_1 + \ldots + p_n = 1$. The base of the legislature is represented by $(s_1, \ldots, s_n)$. Party i is *underrepresented* in the base if $s_i/m < p_i$, and *overrepresented* if $s_i/m > p_i$. In rare cases, a party may be represented exactly in the base with $s_i/m = p_i$.

The augmented legislature is represented by

$$(s_1^*, \ldots, s_n^*), \quad \text{with} \quad s_i \leq s_i^* \leq m \quad \text{for each } i.$$

Let $m^* = s_1^* + \ldots + s_n^*$, so that s_i^*/m^* is the proportion of seats held by party i in the augmented legislature. The aim of PR is to make s_i^*/m^* approximately equal to p_i for each i.

One way to realize this aim is to limit the total seats of each party in the augmented legislature to m, thereby preventing any party from ever exceeding, through additions, the original size of the legislature – and limiting additions to other parties as well. For example, we might seek s_i^* between s_i and m that minimize a measure of the divergence between $(s_1^*/m^*, \ldots, s_n^*/m^*)$ and $(p_1, \ldots, p_n)$, such as $\sum (s_i^*/m^* - p_i)^2$, the sum of squared deviations.

There are, however, a number of problems with this approach. First, it might increase the size of the legislature far beyond m, which not only could prove unwieldy but also thwart the basic idea that single-member districts are primary (e.g., a majority of members in the augmented legislature be elected from the districts). Second, it could change underrepresented parties in the base to overrepresented parties in the augmented legislature, and vice versa, which may be politically unacceptable. Third, it would almost surely lead to anomalies, such as a decrease in proportion in the augmented legislature as a result of a slight increase in p_i for a party, making seats nonmonotonic in votes.

The propensity for this type of phenomenon would depend on the goodness-of-fit criterion used to compare $(s_1^*/m^*, \ldots, s_n^*/m^*)$ to $(p_1, \ldots, p_n)$. Even if additions were monotonic, they could be very sensitive to this criterion, which would almost surely engender considerable controversy. Finally, the minimization approach might encourage parties and their supporters to engage in strategic maneuvering to gain advantage from the particular goodness-of-fit criterion that is used.

Some of these problems can be overcome, or at least made less severe, by an approach that allocates a fixed number of additional seats to the parties on the basis of the election data. However, this approach has problems of its own, not the least of which is the determination and justification of the fixed increment to m. Small fixed increments tend to prevent even a reasonable approximation of PR, whereas large increments exacerbate the problem of dilution of local district choices. Even if an acceptable increment can be agreed upon, as it has been in a few countries, the particular criterion used to allocate the extra seats to the parties may be a point of contention and could well involve anomalies and strategic maneuvering. Moreover, a given fixed increment that might work well in one election could prove quite inadequate in ensuring PR in the next election.

How, for example, is a legislature, which provides for a doubling in size of the original legislature to satisfy PR (like the German Bundestag), to accommodate a three-party national vote split of 40%, 30%, and 30% in which the largest party wins in every district? Some of the 60% voting for the two minority parties cannot be accorded PR if they can get no more than 50% of the seats.

The method described in this paper offers an approach to these problems through a flexible system that is responsive to different situations that might arise without, at the same time, increasing the size of the legislature unduly. It encourages parties to seek the greatest possible turnout, even in districts in which they will obviously win, and it encourages voters to vote their true preferences, thereby completely discouraging both parties and voters from acting strategically. And last, but not least, it attempts to do all this in such a way that PR is at least partially achieved.

3. Rules for Augmentation

In Sect. 1 we advanced three principles, described as rules for augmentation, that might be adopted as guidelines for adding seats to the legislature's base to achieve approximate PR. In this section we shall begin by postulating five rules that in effect provide a precise restatement of our earlier principles, including the one noted at the conclusion of Sect. 1.

These rules apply to all parties i, all vote proportions $(p_1, \ldots, p_n) \geq (0, \ldots, 0)$ for which $\sum p_i = 1$, and all nonnegative integer bases $(s_1, \ldots, s_n)$ and $(s'_1, \ldots, s'_n)$ that have $\sum s_i = \sum s'_i = m$. The primes distinguish one s_i distribution from another – both of which are tacitly assumed to be based on the same national p_i distribution – which we use in Rules 3 and 4 below to state monotonicity conditions for seat additions. As before, "*" denotes the augmented legislature. In Rules 3 and 4, $m'^* = \sum s'^*_i$.

Rule 1 – Catch-up. Additions only for underrepresented. *If* $s_i/m > p_i$, *then* $s^*_i = s_i$.

Rule 2 – Fair Share. Additions give neither overrepresented less, nor underrepresented more, than their fair shares. *If* $s_i/m > p_i$, *then* $s^*_i/m^* \geq p_i$; *if* $s_i/m < p_i$, *then* $s^*_i/m^* \leq p_i$.

Rule 3 – Monotonicity I. Strategyproofness for individual parties: winning does not hurt. *If* $s'_i < s_i$ *and* $s'_j \geq s_j$ *for all* $j \neq i$, *then* $s'^*_i/m'^* \leq s^*_i/m^*$.

Rule 4 – Monotonicity II. Collective strategyproofness for underrepresented parties: winning does not hurt. *If* $s'_i \leq s_i$ *for each* i *for which* $s_i/m < p_i$, *and* $s'_i < s_i$ *for some such* i, *then* $s'^*_i/m'^* \leq s^*_i/m^*$ *for each* i *for which* $s_i/m < p_i$.

Rule 5 – Adjustment. Additions in general never hurt underrepresented parties. *If* $s_i/m < p_i$, *then* $s_i/m \leq s^*_i/m^*$.

Rule 1 says that parties that are overrepresented in the base can obtain no additional seats in the augmented legislature. This has the practical advantage of limiting the increase over the m-seat base, though it could preserve inequities *among* the overrepresented. Rule 1 also underscores the integrity of single-member districts – all parties retain at least these seats – while moving collectively toward greater PR.

Rule 2 says that what has been won at the polls should not be decreased beyond the party's vote proportion in the augmented legislature, nor should parties that lose at the polls be given seats that will raise them above their vote proportions. We think this rule embodies a rather compelling notion of fairness, although slight relaxations might be politically desirable.

Rules 3 and 4 tacitly assume, as we noted earlier, that the same national vote proportions apply to both $(s_1, \ldots, s_n)$ and $(s'_1, \ldots, s'_n)$. Rule 3 says that if one party (i) wins more districts, and none of the other parties does likewise, in going from base $(s'_1, \ldots, s'_n)$ to $(s_1, \ldots, s_n)$, then party i shall not end up with a smaller proportion of seats in the second (unprimed) situation. It is designed to avoid anomalies whereby a party gets a greater proportion of seats in the augmented legislature by losing in more districts; instead, its proportion must be at least as great when it wins. Thus, Rule 3 discourages parties from shaving votes and deliberately losing district elections – they can always do at least as well and sometimes better winning more votes and districts – and also encourages voters to vote for the candidates and parties of their choice.

Rule 4 extends this idea to the collection of underrepresented parties. It says that if one underrepresented party wins more districts in going from s' to s while none does worse, then every one of the underrepresented parties does at least as well in seat proportions in the latter (unprimed) situation. The purpose of Rule 4 is to prevent collusive strategies by blocs of small parties, in which some of them gain by losing in more districts; such strategies are rendered unprofitable by stipulating that winning can never hurt such a party. In the vernacular of economics, it is a collective Pareto-improvement principle for parties that are underrepresented in the base.

Rule 5 complements the second half of Rule 2 by requiring that an underrepresented party shall not end up with smaller proportional representation in the legislature than it would have if no additions were made. Although this seems only fair to such parties, we shall see that it is not always easy to satisfy in seeking to attain PR.

If there are just two parties, then Rules 4 and 5 follow, respectively, from Rule 3, and Rules 1 and 2, and are therefore redundant. In Sect. 4 we show precisely how many additions can be made under Rules 1–3 to the underrepresented party when $n = 2$.

The effect of the rules in the two-party case is illustrated by the following example. Suppose $n = 2$, $m = 8$, and $(p_1, p_2) = (0.2, 0.8)$. In one case, assume party 1 wins one district, party 2 the other seven, so that $(s_1, s_2) = (1, 7)$. Then $s_1/m = 0.125 < p_1$, so party 1 qualifies for additions under Rule 1 – it is underrepresented. However, since $(s_1 + 1)/(m + 1) = 2/9 = 0.222 > p_1$, Rule 2 prevents any additions. Hence, the augmented legislature is the same as the base.

Consider now a second case, in which party 1 wins no districts, so that $(s'_1, s'_2) = (0, 8)$. If only Rules 1 and 2 are used, then party 1 qualifies for *two* seats since this would bring its PR to $2/10 = 0.2$, which is precisely its vote proportion, p_1. Hence, if party 1 can arrange things so that its vote proportion remains approximately the same but it wins no districts, then it comes out better (in case 2; 2 seats out of 10 – losing in all the districts) than in case 1 (1 seat out of 8 – winning in one district).

On the other hand, if Rule 3 is adopted, then party 1 is entitled to only one seat when it wins no districts. This is because two seats would give it a greater proportion (2/10) than it gets when it wins one district (1/8). By contrast, the addition of one seat in case 2 ensures that party 1's proportion (1/9) stays below 1/8, which is its proportion in case 1, as Rule 3 requires.

Thus, in the two-party case, Rule 3, which precludes gains by throwing district elections, limits the ability of the underrepresented party to achieve PR. In other words, the ideal of PR conflicts with the requirement of strategyproofness. As we shall see in Sect. 4, there is a cumulative effect inherent in Rule 3 that builds as we work backward from the most to the fewest districts that could be won by the underrepresented party.

The picture for $n \geq 3$ is much more complex. If all of the rules are imposed, then it may be impossible to make any additions whatsoever, even when some parties are grossly underrepresented. Two examples for three parties illustrate what can happen.

Suppose first that parties 2 and 3 are overrepresented in the base, with p_2 a tiny fraction less than s_2/m, and p_3 considerably less than s_3/m, so that party 1 is significantly underrepresented. Rule 1 says that only party 1 can be augmented. However, if it gets as little as one extra seat, then $s_2/(m + 1)$ will be less than p_2, which violates Rule 2. Thus, our first two rules alone create a problem for PR when $n \geq 3$.

Suppose next that parties 1 and 2 are underrepresented, with p_1 somewhat greater than s_1/m, and p_2 a tiny fraction greater than s_2/m. Then it may be possible to add seats for party 1, but not party 2, that will satisfy Rules 1–4. The problem that any addition for party 2 creates is that it makes $(s_2 + 1)/m^* > p_2$, in violation of Rule 2. However, additions for 1 but not 2 will reduce party 2's PR in the legislature, which violates Rule 5.

These examples for the multiparty case show that, in some circumstances, we must either sacrifice greater PR or else relax some of the rules, including at least one of Rules 1 and 2. The latter course is clearly indicated when PR is a paramount objective, but which rules should be relaxed, and precisely by how much, are debatable matters.

Indeed, the most suitable weakening of the rules for multiparty situations we regard as an open problem for further research, although we shall say a few words about it in Sect. 5. It strikes us as not unlike the problem of apportioning seats to states in the U.S. House of Representatives (and similar bodies), on the basis of decennial censuses, that is analyzed at length in [1]. This work shows that no apportionment method can simultaneously satisfy a few very reasonable conditions: [5] gives an overview of its analysis and questions whether the method that the authors of [1] advocate is clearly superior to all others.

Likewise, in the case of PR, there may be no unequivocably best multiparty compromise for the general case of variable-size legislatures. But the trade-offs in a specific case, consistent with Rules 1–4, are of substantial interest, and this we shall investigate in Sect. 5 to give some flavor of what this kind of analysis entails.

This case involves one overrepresented party and a number of smaller underrepresented parties. The additions allowed by Rules 1–4 seem reasonable in this case, but Rule 5 may be violated in the process. We discuss this and related matters further in Sect. 5 after first analyzing how PR and strategyproofness conflict when there are only two parties.

4. Two-Party Analysis

If $(p_1, p_2) = (s_1/m, s_2/m)$ when there are just two parties, then exact PR holds in the elected legislature, and no changes are called for. The nonexact cases are covered in the following theorem. For notational convenience we shall *let p denote the vote proportion of the underrepresented party*. Although this party will usually be the smaller ($p < 1/2$), it could happen that the party with a majority of the party vote turns out to be the underrepresented party, and in this case $p > 1/2$. Since the aim of the theorem is to show how close we can get to PR without violating Rules 1–3, it includes all the nonexact cases.

Theorem 1. *Suppose one of the two parties has vote proportion p and is underrepresented in the base. Let t be the largest integer that does not exceed* mp, *and let* α_0 *be the largest integer that exceeds neither* $m - t$ *nor* $(mp - t)/(1 - p)$, i.e., $\alpha_0 \leq \min\{m - t, (mp - t)/(1 - p)\}$. *Assume that additions to the legislature's base are governed by Rules* 1, 2, *and* 3, *and suppose that the greatest possible addition to the underrepresented party is made under these rules. Let* s^* *and* m^* *be, respectively, the number of seats held by the underrepresented party in the augmented legislature and the size of this legislature. Then the following conclusions hold for the indicated ranges of p*:

A. $0 < p < 1/2$. α_0 *is either* 0 *or* 1, *and if the underrepresented party wins in* $t - k$ *districts* ($k \geq 0$), *then* $s^* = t + a_0$ *and* $m^* = m + k + \alpha_0$.

B. $p = 1/2$. *$\alpha_0 = 0$ if m is even, $\alpha_0 = 1$ if m is odd and, if the underrepresented party wins $t - k$ districts, then $s^* = t + k + \alpha_0$ and $m^* = m + 2k + \alpha_0$.*

C. $1/2 < p < 1$. *For $k = 1, 2, \ldots$, define the α_k recursively by*

$$\alpha_k = \text{largest integer} \leq \min\left\{m - t, \alpha_{k-1} + \frac{\alpha_{k-1} + t}{m + k - 1 - t}\right\}.$$

If the underrepresented party wins $t - k$ districts, then $s^* = t + \alpha_k$ and $m^* = m + k + \alpha_k$.

A proof of the theorem is given in Appendix 1. We interpret and illustrate its implications in the rest of this section.

The largest number of districts that can be won by a party, without being overrepresented in the base, is t if its vote proportion is p; α_0 is the greatest number of additional seats that it can receive without violating Rules 1 and 2 when it actually wins t districts ($k = 0$ in A, B, and C). Thus, for example, if $m = 100$ and $p = 0.458$, then $t = 45$ and $\alpha_0 = 1$. Although the party's vote proportion in the base is very close to p when it wins 45 districts, it can be given an extra seat with new proportion $46/101 = 0.455$, which is closer to p than is $45/100$ yet does not exceed p (Rule 2).

As shown in part A of the theorem, if $0 < p < 1/2$ – the usual case – then the total seats for the underrepresented party remain fixed at $s^* = t + \alpha_0$, *regardless of how many districts it wins*. However, as its district wins decrease from t back toward 0, more seats need to be add to reach s^*. Hence, the augmented legislature becomes larger. In particular, if the underrepresented party falls k short of winning t districts, then $m^* = m + k + \alpha_0$. Consequently, its proportion in the augmented legislature, namely

$$s^*/m^* = (t + \alpha_0)/(m + k + \alpha_0),$$

decreases as it gets farther away from the target t.

This illustrates the cumulative effect alluded to in the preceding section. Given $s^* = t + \alpha_0$ when the number of district wins is k fewer than t, we can never make $s^* = t + \alpha_0 + 1$, when there are $k + 1$ fewer wins than t, without violating Rule 3, for $(t + \alpha_0 + 1)/(m + k + 1 + \alpha_0) > (t + \alpha_0)/(m + k + \alpha_0)$ whenever $t/m < 1/2$. Since this applies to each k from 0 to $t - 1$, s^* stays fixed but m^* increases as k increases.

Figure 1 shows this graphically. The three lowest curves apply to $p = 0.20$, $p = 0.40$, and $p = 0.49$. We assume that $m = 100$, so $mp = t$ and $\alpha_0 = 0$ for each case illustrated. As the number of districts won by the underrepresented party decreases, its proportion s^*/m^* (vertical scale) in the augmented legislature decreases. At $p = 0.49$, for example, if the party loses every district by a slight margin, it ends up with 49 seats (out of 149), or a representation of about 33% in the augmented legislature.

The effect of additions on the size of the legislature is shown in Fig. 2. For $0 < p < 1/2$ in part A of Theorem 1, the size of the augmented legislature, $m^* + k + \alpha_0$, increases linearly in the shortfall k (below t) of the underrepresented party; the maximum size, $m + t + \alpha_0$, occurs when this party wins no districts.

The dashed curve on the figure indicates likely sizes of the augmented legislature by intersections of this curve with the lines for different values of $p < 1/2$. For example, if $p = 0.20$, the curve intersects the 0.20 line at about (3,117), which would be the actual results if the party with $p = 0.20$ won three of the 100 districts. The augmented legislature would then have 117 seats, for an increase of 17% over the base size of 100

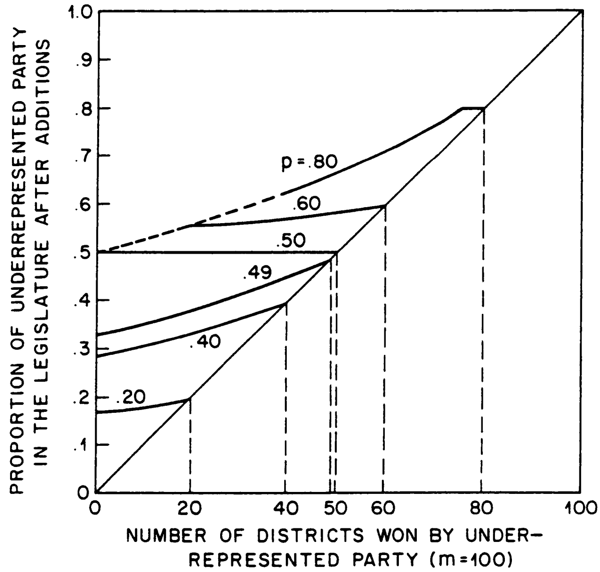

Fig. 1. Underrepresented party proportion in the legislature after additions; under Rules 1, 2 and 3, as a function of districts won and proportion p of total votes cast

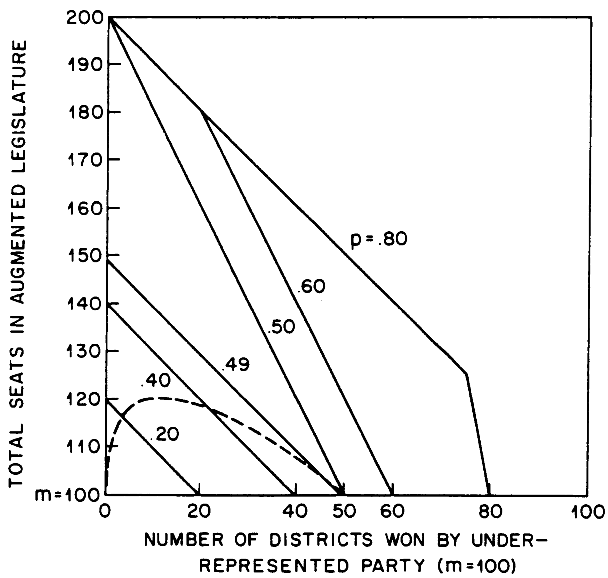

Fig. 2. Total seats in augmented legislature as a function of districts won and vote proportion p for the underrepresented party. Dashed curve suggests probable totals (see text)

seats. The curve is based on the assumption that a party which gets proportion p of the total party vote will win in about $400\,(p^3)\%$ of the districts – i.e., virtually none if p is near 0, up to 50% at $p = 1/2$. There is, of course, nothing magical about the formula, but other sensible estimates of districts won for a given p would probably give qualitatively similar results.

The picture for $0 < p < 1/2$ seems reasonable to us. It offers a compromise between a legislature based solely on wins in the districts and one that is based solely on vote proportions for the different parties. When the underrepresented smaller party does not win its proportional share of the districts, it still receives this number of seats, but not at the expense of subtracting seats that have been won in the district elections by the majority party. Rule 3, which drives the compromise, ensures that both parties and all candidates have every incentive to do as well as they can in every district.

We turn next to part B of Theorem 1. In the unlikely case that p is exactly 1/2, the rules permit additions to the underrepresented party so that exact PR is achieved in the augmented legislature. This can be seen at the point of intersection of the horizontal line for $p = 0.50$ and the vertical axis at $s^*/m^* = 1/2$ in Fig. 1. The effect on size for $p = 0.50$ is shown by the 0.50 line on Fig. 2. As the figures suggest, there is a discontinuity as we pass from $p < 1/2$ to $p = 1/2$, which is caused by the constraint Rule 3 imposes on Rules 1 and 2.

The size line for $p = 0.50$ in Fig. 2 has twice the negative slope of the lines for $p < 0.50$. Although this could mean an increase over base of nearly 100%, the likelihood of such an increase is vanishingly small. In the more likely event that the underrepresented party would win at least 40 of the 100 districts when $p = 1/2$, the size of the legislature would not increase by more than 20%.

We hasten to add that we do not feel that the additions indicated for case B are altogether reasonable. It is true that they achieve exact PR, but in doing so they obliterate any distinction between parties that appears as a result of the district victories. A more reasonable proposal, it seems to us, is to replace the horizontal 0.50 line on Fig. 1 by the limit of the curves for $p < 0.50$ as p approaches 1/2. For this limit, when s districts are won by the underrepresented party,

$$m \text{ odd} \Rightarrow s^* = (m-1)/2,\ m^* = m + (s^* - s);$$

$$m \text{ even} \Rightarrow s^* = m/2,\ m^* = m + (s^* - s).$$

For example, if the underrepresented party wins 45 of the 100 districts when $p = 1/2$, this proposal gives it five (rather than 10) extra seats, the augmented legislature has 105 members, and the proportion of these that belong to the underrepresented party is $50/105 = 0.476$.

The effect of maximum additions to the underrepresented party when it has $p > 1/2$ are shown in the upper part of Fig. 1 and the middle part of Fig. 2. In part C of Theorem 1, α_0 can exceed 1, and the α_k need to be defined recursively since they can change as k changes.

As in part B, the results of part C of Theorem 1 can be challenged. The most serious objection is that, whenever the minority party ($p_i < 1/2$) wins more districts than the majority party ($p_i > 1/2$), the additions to the majority partly indicated in C *must give it control of the legislature*. For example, if $p = 0.60$ and the larger party wins only 47 of the 100 districts, it can receive as many as 26 extra seats under Rules 1–3 ($t = 60$, $\alpha_{13} = 13$, $s^* = 73$) for a proportion of $73/126 = 0.579$ in the augmented legislature.

It is not unreasonable to argue that when a majority of the voters prefers one party over the other, the preferred party should control the legislature, even if it does not win a majority of the districts. Clearly, however, this switch in control could generate considerable controversy, even if it is unlikely to occur, because it involves not only numerical adjustments but also the transfer of power.

5. Multiparty Analysis

Part A of Theorem 1 showed how the achievement of nearly exact PR may be impossible under our principles when there is one smaller underrepresented party in a two-party system. This is caused by Rule 3 constraining Rules 1 and 2.

In the present section, we note ways in which Rules 1–5 can interact in multiparty situations to limit close approximations to PR, even though the rules seem eminently reasonable when considered separately. We shall also state a theorem similar to Theorem 1 for the case in which there is one large party ($p_i > 1/n$) and $n - 1$ smaller parties ($p_i < 1/n$) that are underrepresented. The new theorem uses Rules 1–4 but not Rule 5 – no decreases in proportions in the legislature for underrepresented parties after additions – and determines the maximum additions for this case that are allowed by Rules 1–4. Finally, we suggest priorities for relaxing the rules when approximate PR is a major goal but the rules make realization of this goal impossible.

Some conflicts between Rules 1–5 and PR for the multiparty case were already pointed out in Sect. 3. First, when there is more than one overrepresented party, it may be impossible to make any additions to the base under Rules 1 and 2. Hence, if serious inequities of underrepresented parties are to be alleviated, and Rule 2 is honored for overrepresented parties (proportions in augmented legislature do not fall below vote proportions), it will be necessary to add seats to one or more of the *overrepresented* parties.

Second, if one of the underrepresented parties is just barely underrepresented and Rule 1 is imposed, then a direct clash between Rules 2 and 5 may ensue. In particular, Rule 5 implies that if any underrepresented party gets an additional seat, then every underrepresented party must get at least one additional seat, but this could cause the just-barely underrepresented party to exceed its vote proportion, thus violating Rule 2. Additions of seats to overrepresented parties could help to mitigate this clash, but this in turn raises the possibility of violating the natural dual of Rule 5, i.e., of increasing the seat proportion of an already overrepresented party.

The foregoing conflicts do not involve Rules 3 and 4. Theorem 2, which is proved in Appendix 2, illustrates the sequential effects these rules have on achieving approximate PR.

Theorem 2. *Suppose each of the first $n - 1$ of $n \geq 3$ parties has $p_i < 1/n$ and is underrepresented in the base. For each $i \leq n - 1$, let t_i be the largest integer strictly less than mp_i, and let S denote the set of all vectors $(\beta_1, \ldots, \beta_{n-1})$ of nonnegative integers that satisfy*

$$t_i + \beta_i \leq m \quad \textit{and} \quad \frac{t_i + \beta_i}{m + \sum \beta_j} \leq p_i \quad \textit{for } i = 1, \ldots, n - 1.$$

Then $\beta_1 + \ldots + \beta_{n-1}$ is maximized by a unique vector in S. Let this maximizing vector be $(\beta_1^*, \beta_2^*, \ldots, \beta_{n-1}^*)$. Assume that additions to the base are governed by Rules 1 through 4 for all $(s_1, \ldots, s_{n-1}) \leq (t_1, \ldots, t_{n-1})$, and suppose that the greatest total additions to the underrepresented parties are made under these rules. Then party i gets $t_i + \beta_i^*$ seats in the augmented legislature for each $i \leq n-1$, party n gets $m - (s_1 + \ldots + s_{n-1})$ seats – the number of districts it wins – and the size of the augmented legislature is

$$m + \sum_{i=1}^{n-1} (t_i + \beta_i^* - s_i)$$

for each base $(s_1, \ldots, s_{n-1}) \leq (t_1, \ldots, t_{n-1})$.

The example at the end of Sect. 1 shows that all the β_i can be 0, in which case each smaller party gets t_i seats. That example also shows that the solution in Theorem 2 can violate Rule 5. Party 4, with 28 district seats and $p_4 = 0.293$, has $s_4^*/m^* = 29/108 = 0.269$, which is below its base proportion of 0.28, as we noted earlier.

This is caused by the fact that each smaller party gets $t_i + \beta_i^*$ seats, regardless of how many districts it wins, assuming they all win no more than t_i districts. It may, therefore, appear that each such party has no incentive to win districts, only to get as large a proportion p_i of the party vote as possible. But this is not so. In particular, if any of the smaller parties beats the large party in more districts, then the size of the augmented legislature will be smaller and the proportions of the smaller parties in that legislature will be larger. For example, if parties 4 and 5 in the example at the end of Sect. 1 had won, respectively, 29 and 41 districts instead of 28 and 42, i.e., the smaller party gains relative to the larger one, then the augmented legislature would have $m^* = 107$ instead of 108, with $s_4^*/m^* = 29/107 = 0.271$ – compared to 0.269 in the example. (Note that 0.271 is below party 4's new base proportion of 0.29, again violating Rule 5.)

The addition of Rule 4 to Rules 1–3 in Theorem 2 can make a slight difference. If only the first three rules had been used in the theorem, then the t_i and β_i^* would not change, but even more additions could occur for some $(s_1, \ldots, s_{n-1}) \leq (t_1, \ldots, t_{n-1})$. For example, if $n = 3$, $m = 100$ and $(p_1, p_2, p_3) = (0.301, 0.306, 0.393)$, then no additions can be made under Rules 1 and 2 when $(s_1, s_2, s_3) = (30, 30, 40)$. However, with $(s_1', s_2', s_3') = (29, 30, 41)$, Rules 1–3 allow an additional seat to be added to each of the first two parties, to give

$$s'^* = (30, 31, 41), \quad m'^* = 102.$$

It is easily checked that this does not violate Rule 2, and it does not violate Rule 3 for either party 1 $(30/102 < 30/100)$ or party 3 $(41/102 > 40/100)$. However, it violates Rule 4 for party 2 since $31/102 > 30/100$.

Although it is difficult to predict precisely the expected size of the legislature if the solution of Theorem 2 is used, an increase over base in the range of 5%–20% seems reasonable. For example, if a large party with about 40% of the total party vote wins in a narrow majority of the districts, then the increment would be about 10–15%.

Because the addition of no seats in the multiparty case satisfies all our rules, they are logically consistent. However, as we have seen, strict adherence to the rules may make close approximations of PR impossible. It is, therefore, desirable to explore relaxations of Rules 1–5 that are more congruent with the goal of PR. We shall make a few suggestions in this regard without proposing a complete resolution of the matter.

For discussion purposes, we divide the rules into three main groups, which are listed in approximate order of our own sense of importance:

Group 1. Fairness Rules: Rules 2 and 5. Rules in this group compare each possible base $(s_1, \ldots, s_n)$ with its augmented seat distribution $(s_1^*, \ldots, s_n^*)$ and the vote proportions $(p_1, \ldots, p_n)$.

Group 2. Incentive-Compatible Rules: Rules 3 and 4. Rules in this group compare situations for different pairs of bases, $(s_1, \ldots, s_n)$ and $(s'_1, \ldots, s'_n)$. They are designed to encourage parties to do as well as possible in all districts and to discourage strategic maneuvers.

Group 3. Basic Addition Rules: Rule 1. These rules identify which parties can receive additional seats and may also limit the numbers of seats that can be added to individual parties or to the original legislature.

Since Rules 2 and 5 are such important fairness conditions, we feel that only small deviations from them could be tolerated. For example, Rule 2 could be relaxed for underrepresented parties by allowing s_i^*/m^* to be slightly greater than p_i, say with s_i^* no larger than the integer part of $p_i m^* + 1$ when m is substantially bigger than n. Any relaxation of Rule 5 may be more problematic, and in fact may be unnecessary if the suggested relaxation of Rule 2 is adopted.

In Group 2, we regard Rule 3 as more basic than Rule 4 and would not suggest any change in Rule 3. The elimination of Rule 4 may not be too serious, for we suspect that it would hold to a fair approximation in most reasonable augmentation schemes that satisfy Rule 3. The following weakening of Rule 4 captures much of its intention, yet it would be easier to satisfy in practice:

Rule 4′. If $s'_i \le s_i$ for each i for which $s_i/m < p_i$, and $s'_i < s_i$ for some such i, then $(\sum s_i'^*)/m'^* \le (\sum s_i^*)/m^*$, where the sum extends over the underrepresented parties.

This says that a Pareto-improvement in the seats won by the underrepresented parties shall not decrease their *collective* PR in the augmented legislature. It is clearly implied by Rule 4, whose conclusion is $s_i'^*/m'^* \le s_i^*/m^*$ for each underrepresented party taken separately.

Rule 1 seems least essential, except for its effect on limiting increments to the base. It is also one of the greatest constraints on achieving PR in some multiparty cases. We therefore suggest that it be relaxed to allow additions to overrepresented parties – within certain bounds – especially when needed additions to underrepresented parties would simultaneously decrease the seat proportion of an overrepresented party below its vote proportion, in violation of Rule 2. That is, because we think increases to the base should be kept within tolerable limits, we would suggest that Rule 1 be relaxed only to the extent needed to satisfy Rule 2 for overrepresented parties.

In summary, we feel that the following set of rules provides a sensible basis for exploring the multiparty case further:

Rule 1: Modify to allow additions to overrepresented parties when Rule 2 for overrepresented parties would otherwise be violated;

Rule 2: Modify by allowing s_i^*/m^* to exceed p_i by a small amount for an underrepresented party, but only when this is needed to make reasonable additions to other underrepresented parties (and retain the original version of Rule 2 for overrepresented parties);

Rule 3: Retain intact;

Rule 4: Replace by Rule 4′;

Rule 5: Retain intact.

As stated here, our proposed modifications of Rules 1 and 2 are somewhat imprecise since we do not feel confident at this time about the most suitable details. We invite others to consider the matter. It is possible, of course, that there are conflicts between the modified set and PR that we have not foreseen, in which case further relaxations may be needed.

We have considered a few augmentation schemes for the general multiparty case that appear to satisfy the spirit if not the letter of our set of modified rules, but we defer judgment at this time on an "ideal" scheme. However, as a conclusion to this section, we offer some thoughts on this matter and note problems that can arise.

Because we believe approximating PR is desirable, we wish to give as many additional seats to the underrepresented parties as possible under the modified Rules 2–5. One approximate way to do this, guided by the results of Theorems 1 and 2, is to increase the underrepresented party's seats up to the largest integers that do not exceed their $p_i m$ bounds, and then to add as few additional seats as necessary under modified Rule 2 to satisfy Rule 5 if it is not already satisfied by the initial additions. Alternatively, one could use Theorem 1 directly, treating the collections of underrepresented parties and overrepresented parties as two aggregate parties, to determine the total number of seats that can be added to the underrepresented parties. This total, or something close to it, would then be allocated to these parties so as to satisfy Rule 5 and the modified Rule 2. These two procedures yield about the same additions.

If these additions satisfy Rule 2 for the *overrepresented* parties, nothing more needs to be done. However, if they reduce s_i/m^* for one or more overrepresented parties below its p_i, then to satisfy Rule 2 for these parties we must either add seats to them under modified Rule 1, or take away some of the seats added to the underrepresented parties in the first stage of the process.

The choice here seems problematic. If seats added to the underrepresented parties are selectively subtracted to satisfy Rule 2 for the overrepresented parties, it would seem best to subtract as few seats as possible for this purpose without violating Rule 5 for the underrepresented parties. On the other hand, if seats are added to overrepresented parties, we would add as few as necessary to satisfy Rule 2 for those parties. In the latter case, the further additions to some overrepresented parties could cause violations of Rule 5 for underrepresented parties since their initially augmented seat proportions will be decreased by these additions. This in turn could cause new problems for Rule 2 for overrepresented parties when further additions are made to correct for violations of Rule 5 for underrepresented parties. In a knife-edge situation with one barely underrepresented party and another barely overrepresented party, successive iterations of the add-on method might drive the "solution" to an unacceptable extreme.

We illustrate these ideas with data from the June 1983 election for the British House of Commons. Three parties won all but 21 of the 650 seats for the singlemember

Table 1. Based on June, 1983 election for the British House of Commons

	Vote percent $[100 p_i]$	Seats won $[s_i]$	First increase	Second increase	Final seat percent
Conservative	42.4	397	397	397	48.6
Labor	27.6	209	209	226	27.7
Alliance	25.3	23	164	164	20.1
Other	4.7	21	30	30	3.7
	100%	650	800	817	100%

districts. The party vote and seats won by these three and by the others (grouped together) are shown in the first two columns of Table 1. The most striking result is that the Alliance received 25.3% of the vote but won only 3.5% of the seats. First additions to the underrepresented parties (Alliance, Other) under our modified rules yields the augmented legislature in the third column, with a total of 800 seats. Here Alliance goes from 23 to 164 seats $[(0.253)650 = 164.45]$, and Other from 21 to 30 seats $[(0.047)650 = 30.55]$. These increases satisfy the modified rules except Rule 2 for Labor, whose seat proportion $(209/800 = 0.261)$ falls below its vote proportion (0.276). The add-on method described in the preceding paragraph is then used to add seats to Labor (initially overrepresented) to increase its seat proportion above its vote proportion. This second increase gives an 817-seat legislature, whose seats and seat percents are shown in the final two columns of the table. Since this satisfies Rules 2 and 5, no further changes are made. The 26% increase in the 817-seat legislature over the 650-seat base is caused mainly by the Alliance's large initial underrepresentation.

There is a further problem with additions to overrepresented parties in that it may be impossible to complete such additions before one runs into the upper bound of m seats per party. Suppose, for example, that there are 100 districts, three parties, $(p_1, p_2, p_3) = (1/9, 2/3, 2/9)$, and $(s_1, s_2, s_3) = (33, 67, 0)$. Only party 3 is underrepresented. Since it wins no seats, the procedure suggested above gives it 22 seats, since $(2/9)(100) = 22.222$. The augmented legislature now has 122 seats, but this puts party 2 below p_2 in seat proportion since $67/122 < 2/3$.

If we follow the subtraction procedure and do not add any seats to party 2, then party 3 ends up with no seats at all since 67/101 is already less than 2/3 (p_2). Because this seems unacceptable, we add seats to party 2 to bring its seat proportion up to 2/3. However, given the initial addition of 22 seats to party 3, this is impossible: even if party 2 gets 33 additional seats, bringing its total to 100, its seat proportion will be only $100/(122 + 33) = 0.645$, which is less than 2/3.

Thus, while both the subtraction and additions-to-overrepresented-parties proposals may have seemed sensible for correcting violations of Rule 2 for overrepresented parties, neither is suitable in the example, and another procedure is called for. The only obvious alternative is to do both simultaneously, i.e., to add seats to party 2 and subtract from the initial additions to party 3. If, in fact, party 2 is given 33 additional seats, then the most that party 3 can get without violating Rule 2 for party 2 is the largest integer k for which $100/(133 + k) \geq 2/3$, which is $k = 17$. Consequently, the best that can be done for party 3 when Rule 2 is honored for party 2 is the augmented legislature $(s_1^*, s_2^*, s_3^*) = (33, 100, 17)$, with seat proportions (0.220, 0.667, 0.113).

Although the preceding example is admittedly concocted for purposes of illustration, we hope that it suggests the complexity of the problem of designing a reasonable augmentation procedure for the general multiparty case.

6. Conclusions

In this paper we have proposed a set of rules for additions of seats to parties that are underrepresented in a legislature as the result of elections in single-member districts. The purpose of the additions is to achieve approximate PR in the legislature, according to the nationwide party vote, without undermining the integrity of the districts. That is, we want the winner in each district to be assured a seat in the legislature at the same time that the additions are kept within reasonable bounds.

We have shown that our rules work fairly well in the two-party case, especially when the underrepresented party has the smaller party vote, although they can limit the extent to which PR is realized. A similar conclusion – with adjustments for Rule 5 – applies in the multiparty case when there is one large party and a number of smaller, underrepresented parties.

The general multiparty situation, however, is a good deal more complex. To approximate PR in this case, it is clear that the rules must be relaxed; we have suggested certain modifications that seem to us most palatable. Even so, it is unclear what augmentation procedure is best suited for this case. We hope that further research will clarify this challenging problem.

Appendix 1

Given the conditions of Theorem 1, including $t \le mp \le t + 1$ and α_0 as the largest integer that exceeds neither $m - t$ nor $(mp - t)/(1 - p)$, suppose first that $0 < p < 1/2$ (part A). If the underrepresented party – hereafter denoted UP – wins t districts, then it can get α_0 additional seats but no more without violating Rule 2; otherwise, $(t + \alpha_0)/(m + \alpha_0) > p$. It is easily seen that $\alpha_0 = 1$ is possible in some cases, but $\alpha_0 \ge 2$ is not since $2 \le (mp - t)/(1 - p)$ implies $2 < 1/(1 - p)$, which implies $p > 1/2$, contrary to the conditions of part A. Thus $s^* = t + \alpha_0$ and $m^* = m + a_0$ for the most additions when UP wins t districts.

We complete the proof of part A by an induction argument. Suppose the stated result ($s^* = t + \alpha_0$, $m^* = m + k + \alpha_0$) holds when UP wins $t - k$ districts. Consider what happens when it wins only $t - (k + 1)$ districts. The proposed solution for this case ($s^* = t + \alpha_0$, $m^* = m + (k + 1) + \alpha_0$) satisfies Rules 1–3; this is the best we can do, for suppose to the contrary that ($s^* = t + \alpha_0 + \beta$, $m^* = m + (k + 1) + \alpha_0 + \beta$) with $\beta \ge 1$ is feasible under the rules. Then Rule 3 requires that

$$\frac{t + \alpha_0 + \beta}{m + k + 1 + \alpha_0 + \beta} \le \frac{t + \alpha_0}{m + k + \alpha_0},$$

where the right-hand side is UP's proportional representation when it wins $t - k$. This inequality can be rewritten as $\beta/(1 + \beta) \le (t + \alpha_0)/(m + k + \alpha_0)$, whose right-hand side must not exceed p by Rule 2. Therefore $\beta/(1 + \beta) < 1/2$, hence $\beta < 1$, contradicting $\beta \geq 1$. Since the proposed solution holds when $k = 0$, it holds by induction for all $k \geq 0$.

Suppose next that $p = 1/2$ (part B). If m is even, then $t = m/2$ and $\alpha_0 = 0$. If m is odd, then $t = (m - 1)/2$ and $\alpha_0 = 1$. In either case, and for every feasible $k \geq 0$ for which the UP wins $t - k$ districts, the proposed solution ($s^* = t + k + \alpha_0$, $m^* = m + 2k + \alpha_0$) gives both parties PR of 1/2 in the augmented legislature. This solution obviously satisfies the rules and, because it is the best possible under Rule 3, the proof of part B is complete.

Finally, suppose UP has $1/2 < p < 1$. Let the α_k be as defined in part C of Theorem 1. By definition, α_0 is the most seats that can be added to UP without exceeding p, and without exceeding the total number of districts (m), when UP wins t districts. If it wins $t - 1$ districts, and its total number of seats after addition is $t + \alpha_1$, then we require $t + \alpha_1 \leq m$ and, for Rule 3 (or Rule 2 if $mp = t$), we also require

$$\frac{t + \alpha_1}{m + 1 + \alpha_1} \leq \frac{t + \alpha_0}{m + \alpha_0}.$$

The latter inequality can be rewritten as

$$\alpha_1 \leq \alpha_0 + \frac{\alpha_0 + t}{m + k - 1 - t} \quad (k = 1).$$

Hence, with α_1 defined as the largest integer which exceeds neither $m - t$ nor $\alpha_0 + (\alpha_0 + t)/(m - t)$, we can add α_1 seats to UP, but no more, when it wins $t - 1$. We now repeat this procedure when it wins $t - 2$ instead of $t - 1$, then $t - 3$ instead of $t - 2$, and so forth. At each step, α_k is the most seats that can be added to UP when it wins $t - k$ districts.

Appendix 2

Given the conditions of Theorem 2, suppose first that each of the first $n - 1$ parties wins as many districts as possible, namely t_i, while being underrepresented in the base. Each $(t_1 + \beta_1, \ldots, t_{n-1} + \beta_{n-1},\ m - t_1 - \ldots - t_{n-1})$ for $(\beta_1, \ldots, \beta_{n-1}) \in S$ is a feasible augmented legislature according to the definition of S and Rules 1 and 2. Moreover, if $(\beta_1, \ldots, \beta_{n-1})$ and $(\beta'_1, \ldots, \beta'_{n-1})$ are in S, then so is $(\max\{\beta_1, \beta'_1\}, \ldots, \max\{\beta_{n-1}, \beta'_{n-1}\})$ since the sum of the components in the latter vector exceeds or equals each of $\sum \beta_i$ and $\sum \beta'_1$. (Therefore the $\leq p_i$ inequalities must hold for the max vector.) It follows that there is one and only one vector in S that maximizes $\sum \beta_i$. Call it $(\beta^*_1, \beta^*_2, \ldots, \beta^*_{n-1})$.

Since we are interested in making the greatest number of additions to the base under the rules, it follows that, when the base has $(s_1, \ldots, s_{n-1}) = (t_1, \ldots, t_{n-1})$, the augmented legislature will have $s^* = (t_1 + \beta^*_1, \ldots, t_{n-1} + \beta^*_{n-1},\ m - t_1 - \ldots - t_{n-1})$ and size $m^* = m + \sum \beta^*_i$.

Consider next the general case of $(s_1, \ldots, s_{n-1}) \leq (t_1, \ldots, t_{n-1})$ and the proposed solution of the theorem given by

$$s^* = (t_1 + \beta^*_1, \ldots, t_{n-1} + \beta^*_{n-1}, m - s_1 - \ldots - s_{n-1})$$

with $m^* = m + \sum(t_i + \beta_i^* - s_i)$. This solution obviously satisfies Rules 1 and 2. To show that it satisfies Rule 3 for the restricted domain of bases (underrepresented for all $i \leq n-1$), suppose first that $s_1' < s_1$ and $s_j' \geq s_j$ for all $j \geq 2$. Rule 3 requires

$$\frac{t_1 + \beta_1^*}{m + \sum_{i \leq n-1} (t_i + \beta_i^* - s_i')} \leq \frac{t_1 + \beta_1^*}{m + \sum_{i \leq n-1} (t_i + \beta_i^* - s_i)},$$

which reduces to $\sum_{i=1}^{n-1} (s_i - s_i') \geq 0$, or, since $\sum_{i=1}^{n} s_i = \sum_{i=1}^{n} s_i' = m$, to $s_n' - s_n \geq 0$, and this is true by hypothesis. By analogy, the same result holds for each $i \leq n-1$, and it is easily seen that Rule 3 holds when $s_n' < s_n$ and $s_j' \geq s_j$ for all $j < n$. Therefore Rule 3 is satisfied by the proposed solution, and Rule 4 is easily seen to hold.

To prove that this solution allows the most total additions to the base, we proceed by induction. Our induction hypothesis is that, given the most total additions at $(t_1, \ldots, t_{n-1})$, the most total additions at $(s_1, \ldots, s_{n-1})$ is uniquely specified by s^* in the preceding paragraph. Assuming that some $s_i > 0$, say $s_1 > 0$ for definiteness, we wish to show that the proposed solution at $(s_1 - 1, s_2, \ldots, s_{n-1})$ uniquely allows the most additions under Rules 1–4. If this is true in every case, it follows by induction that the theorem is true since the proposed solution at $(t_1, \ldots, t_{n-1})$ is uniquely maximizing in additions.

Contrary to the claim that the proposed solution uniquely allows the most additions at $(s_1 - 1, s_2, \ldots, s_{n-1})$, suppose a different vector of as many or more additions also satisfies the rules. Then, as in the first paragraph of this proof, the component-by-component maximum of the two also satisfies the rules, and especially Rule 4. That is, there are nonnegative integers δ_i with $\delta_i \geq 1$ for some $i \leq n-1$ such that (for Rule 4, by comparison with $(s_1, \ldots, s_{n-1})$), for each $i \leq n-1$,

$$\frac{t_i + \beta_i^* + \delta_i}{m + \sum(t_i + \beta_i^* - s_i) + 1 + \sum \delta_i} \leq \frac{t_i + \beta_i^*}{m + \sum(t_i + \beta_i^* - s_i)} \leq p_i < \frac{1}{n}.$$

The first inequality here can be rewritten as

$$\delta_i [m + \sum(t_i + \beta_i^* - s_i)] \leq (1 + \sum \delta_i)(t_i + \beta_i^*).$$

Moreover, the last two inequalities give $(t_i + \beta_i^*) < [m + \sum(t_i + \beta_i^* - s_i)]/n$, and therefore $n\delta_i < 1 + \sum \delta_i$. Since the δ_i are integers, it follows that

$$n\delta_i \leq \sum \delta_i \quad \text{for} \quad i = 1, \ldots, n-1,$$

and summation over i then gives $n \sum \delta_i \leq (n-1) \sum \delta_i$, which is false if $\sum \delta_i > 0$. Thus we obtain a contradiction, and conclude that the proposed solution uniquely maximizes the additions possible at $(s_1 - 1, s_2, \ldots, s_n)$.

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